# Structure Function Representation of Multi-Directional Heat-Flows

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***Abstract* — A relation is established between the structure function of the RC transmission line modeling a one-port passive distributed thermal network and the spatial distribution of thermal properties in heat diffusion problems with generic multi-directional heat-flows.**

## 1 Introduction

Distributed thermal networks are widely used for modeling heat conduction problems in components and packages. Structure functions, originally introduced by V. Szekely et al. [1], are means for characterizing the responses of one-port distributed thermal networks in terms of equivalent RC transmission lines. In fact, as recently precised in [8], the response of a one-port passive distributed thermal network modeling a generic thermal problem coincides with the response of a proper RC transmission line defined by a structure function.

Since their introduction, structure functions have been used for inferring on the spatial distributions of thermal properties in components and packages [1–4]. For a one-dimensional thermal problem in which power is injected at one boundary and temperature rise is measured at the same boundary, a one-port passive distributed thermal network naturally arises. In this case, the cumulative resistance and capacitance of the RC transmission line are exactly the cumulative thermal resistance and capacitance from the boundary at which power is injected. In this case, hereafter referred to as one-directional heat flow, by determining the structure function from the thermal response of the distributed thermal network, precise information on the spatial distribution of thermal properties is recovered [13].

No procedure is reported in literature for exactly determining a structure function from a thermal response. Moreover in practice, this technique is used in applications which are only approximated by one-directional heat-flows. The more inaccurate is this approximation, the more inaccurate is the recovered information on the spatial distribution of thermal properties. That is because no exact interpretation of the structure function is available. The situation is even more critical for a generic thermal problem, hereafter referred to as multi-directional heat-flow. For such a thermal problem, the structure function can still be associated to a one-port passive distributed thermal network, as precised in [8]. However the relation between the structure function and the spatial distribution of thermal properties in the thermal problem is not known.

In this paper, which extends the results in [5], the structure function of a one-port passive distributed thermal network modelling a multi-directional heat-flow is considered. A procedure for exactly determining a structure function from a thermal response is shown. Moreover an exact relation between the structure function of the equivalent RC transmission line on one hand and the spatial distribution of thermal properties in the thermal problem on the other hand is established. As a result a physical interpretation of the structure function is obtained.

Precisely, firstly the problem of relating the structure function of an RC transmission line to the spatial distribution of material properties of the heat diffusion problem is transformed into the equivalent problem of relating the structure function of an LC transmission line to the spatial distribution of material properties of a wave propagation problem. Secondly the relation between the structure function and the response of such LC transmission line is outlined. Lastly a family of wave fronts and a family of rays are naturally associated to the wave propagation problem. The structure function is related to the thermal properties along rays between wave-fronts. Using this result, a strategy is proposed for naturally exploiting structure functions in practical applications.

The rest of this paper is organized as follows. In section 2 the definition of structure function is recalled. In section 3 the companion wave propagation problem is introduced. A procedure for determining the structure function from the wave-propagation problem is outlined in section 4 and applied in 5. Wave-fronts and rays of the wave propagation problem and the relation between structure function and spatial distribution of material properties is shown in section 6 and applied in section 7.

## 2 RC Transmission Line Representation of One-Port Passive Distributed Thermal Network

We refer to a generic component or package that extends in a bounded spatial region $\Omega$. As is well known, the relation between the generated power density $G(\mathbf{r},t)$ and the temperature rise distribution $u(\mathbf{r},t)$ with respect to ambient temperature and the heat flux density $\mathbf{q}(\mathbf{r},t)$ is ruled by the First Principle of Thermodynamics and by Fourier's law as follows

$$\nabla \cdot \mathbf{q}(\mathbf{r},t)) + c(\mathbf{r})\frac{\partial v}{\partial t}(\mathbf{r},t) = G(\mathbf{r},t), \tag{1}$$

$$\mathbf{q}(\mathbf{r},t) = -k(\mathbf{r})\nabla u(\mathbf{r},t), \tag{2}$$

in which $c(\mathbf{r})$ is the volumetric heat capacity and $k(\mathbf{r})$ is the thermal conductivity. These equations are completed by the conditions on the boundary $\partial\Omega$ of $\Omega$ and by the initial condition for the temperature rise $u(\mathbf{r},t)$. The boundary conditions, assumed of Robin's type, are

$$\mathbf{q}(\mathbf{r},t) \cdot \nu(\mathbf{r}) = h(\mathbf{r})u(\mathbf{r},t), \tag{3}$$

in which $h(\mathbf{r})$ is the heat transfer coefficient and $\boldsymbol{\nu}(\mathbf{r})$ is the unit vector outward normal to $\partial\Omega$. The initial condition is assumed to be zero

$$u(\mathbf{r},0) = 0. \tag{4}$$

A one-port passive distributed thermal network is defined as in [8–10], by introducing the power $P(t)$ and the temperature rise $T(t)$ measured at its port. The power $P(t)$ determines the power density $G(\mathbf{r},t)$ as

$$G(\mathbf{r},t) = g(\mathbf{r})P(t) \tag{5}$$

in which $g(\mathbf{r})$ is a function of support $\Sigma$. The temperature rise $T(t)$ is

$$T(t) = \int_\Omega g(\mathbf{r})u(\mathbf{r},t)\, d\mathbf{r}. \tag{6}$$

The relation between the power $P(t)$ and the temperature rise $T(t)$ is represented, in the time domain, by a power impulse thermal response $z_{RC}(t)$ and, in the complex angular frequency domain, by a thermal impedance function $\mathsf{Z}_{RC}(s)$.

As proved in [8], the response of a one-port passive distributed thermal network is the short circuit input response of a passive RC transmission line of total resistance $R_0 = \mathsf{Z}_{RC}(0)$. Any definition of the port temperature $T(t)$ different from Eq. (6) in general leads to an active distributed thermal network, which clearly is not equivalent to a passive RC transmission line as shown in Fig. 1. The non-

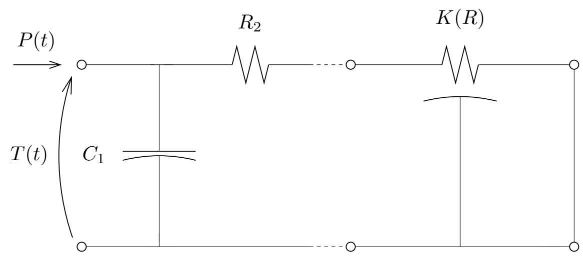


Figure 1: Equivalent passive RC transmission line.

singular RC transmission line is ruled by equations

$$P(x,t) = -k(x)\frac{\partial T}{\partial x}(x,t),$$
$$\frac{\partial P}{\partial x}(x,t) = -c(x)\frac{\partial T}{\partial t}(x,t)$$

in which $k(x)$ and $c(x)$ are the thermal conductivity and volumetric heat capacity at coordinate $x$ along the line, $P(x,t)$ and $T(x,t)$ are the power and temperature rise at coordinate $x$ along the line. Different choices of the $x$ coordinate correspond to different $k(x)$ and $c(x)$ and to different representations of the non-singular RC transmission line. Two representations are hereafter considered. In the *travel time* representation [14, 15], the coordinate $x = \tau$ is chosen is such a way that $k(x) = c(x) = A(\tau)$ so that

$$P(\tau,t) = -A(\tau)\frac{\partial T}{\partial \tau}(\tau,t),$$
$$\frac{\partial P}{\partial \tau}(\tau,t) = -A(\tau)\frac{\partial T}{\partial t}(\tau,t)$$

In the *structure function* representation [6, 7], the coordinate $x = R$ is chosen in such a way that $k(x) = 1$ and $c(x) = K(R)$ so that

$$P(R,t) = -\frac{\partial T}{\partial R}(R,t),$$
$$\frac{\partial P}{\partial R}(R,t) = -K(R)\frac{\partial T}{\partial t}(R,t).$$

Thus $R$ is the cumulative resistance along the line and

$$C(R) = \int_0^R K(R)\, dR$$

is the cumulative capacitance.

Such representations are equivalent. In fact from the travel time representation the structure function representation is recovered as follows

$$R = \int_0^\tau \frac{d\tau}{A(\tau)}, \tag{7}$$

$$K(R) = A^2(\tau), \tag{8}$$

$$C(R) = \int_0^\tau A(\tau)\, d\tau. \tag{9}$$

Similarly from the structure function representation the travel time representation is recovered as follows

$$\tau = \int_0^R \sqrt{K(R)}\, dR,$$
$$A(\tau) = \sqrt{K(R)}.$$

## 3 Companion Wave Propagation Problem

The heat diffusion problem of Eqs. (1)-(4) and the one-port passive distributed thermal network of Eqs. (5), (6) can be formally associated to a companion wave propagation problem and to a companion one-port passive distributed network. Precisely the wave propagation problem in $\Omega$ is,

$$\frac{\partial \mathbf{j}}{\partial t}(\mathbf{r}, t) = -k(\mathbf{r})\nabla v(\mathbf{r}, t) \tag{10}$$

$$\nabla \cdot \mathbf{j}(\mathbf{r}, t)) + c(\mathbf{r})\frac{\partial v}{\partial t}(\mathbf{r}, t) = G(\mathbf{r}, t), \tag{11}$$

with boundary conditions, on $\partial\Omega$,

$$\frac{\partial \mathbf{j}}{\partial t}(\mathbf{r}, t) \cdot \boldsymbol{\nu} = h(\mathbf{r})v(\mathbf{r}, t), \tag{12}$$

and with initial conditions, in $\Omega$,

$$v(\mathbf{r}, 0) = 0, \quad \mathbf{j}(\mathbf{r}, 0) = 0. \tag{13}$$

The companion one-port passive distributed network is defined by introducing the current $I(t)$ and the voltage $V(t)$ measured at its port, as follows. The current $I(t)$ determines $G(\mathbf{r}, t)$ as

$$G(\mathbf{r}, t) = g(\mathbf{r})I(t). \tag{14}$$

The voltage $V(t)$ is

$$V(t) = \int_\Omega g(\mathbf{r})v(\mathbf{r}, t)\, d\mathbf{r}. \tag{15}$$

The relation between the current $I(t)$ and the voltage $V(t)$ is represented, in the time domain, by an impulse response $z_{LC}(t)$ and, in the complex angular frequency domain, by an impedance function $\mathsf{Z}_{LC}(s)$. Since

$$\mathsf{Z}_{LC}(s) = s\,\mathsf{Z}_{RC}(s^2),$$

the response of this one-port distributed network is the short circuit input response of a passive LC transmission line obtained from the passive RC transmission line of Fig. 1 by substituting the resistors with inductors, as shown in Fig. 2. This passive LC transmission line is in general singular since it consists of the cascade connection of a

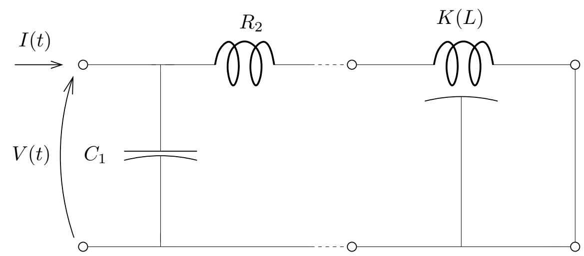


Figure 2: Equivalent passive LC transmission line.

lumped LC ladder and of a non-singular LC transmission line ruled by equations

$$\frac{\partial I}{\partial t}(x, t) = -k(\tau)\frac{\partial V}{\partial x}(x, t),$$
$$\frac{\partial I}{\partial x}(x, t) = -c(x)\frac{\partial V}{\partial t}(x, t)$$

in which $I(x, t)$ and $V(x, t)$ are the current and voltage at coordinate $x$ along the line. Different choices of the $x$ coordinate correspond to different $k(x)$ and $c(x)$ and to different representations of the non-singular LC transmission line. Two representations are hereafter considered. In the travel time representation [14, 15] the coordinate $x = \tau$ is chosen in such a way that $k(x) = c(x) = A(\tau)$ so that

$$\frac{\partial I}{\partial t}(\tau, t) = -A(\tau)\frac{\partial V}{\partial \tau}(\tau, t), \tag{16}$$

$$\frac{\partial I}{\partial \tau}(\tau, t) = -A(\tau)\frac{\partial V}{\partial t}(\tau, t) \tag{17}$$

In the *structure function* representation [3], the coordinate $x = L$ is chosen in such a way that $k(x) = 1$ and $c(x) = K(L)$ so that

$$\frac{\partial I}{\partial t}(L, t) = -\frac{\partial V}{\partial L}(L, t),$$
$$\frac{\partial I}{\partial L}(L, t) = -K(L)\frac{\partial V}{\partial t}(L, t).$$

Thus $L$ is the cumulative inductance along the line and

$$C(L) = \int_0^L K(L)\, dL$$

is the cumulative capacitance. Such representations are equivalent. In fact from the travel time representation the structure function is recovered as follows

$$L = \int_0^\tau \frac{d\tau}{A(\tau)},$$
$$K(L) = A^2(\tau),$$
$$C(L) = \int_0^\tau A(\tau)\, d\tau.$$

Similarly from the structure function representation, the travel time representation is recovered as follows

$$\tau = \int_0^L \sqrt{K(L)}\, dL,$$
$$A(\tau) = \sqrt{K(L)}.$$

The spatial distributions of material properties $k(\mathbf{r})$ and $c(\mathbf{r})$ are common to the heat diffusion problem and to the companion wave propagation problem. Similarly the structure function representation is common to the one-port passive distributed thermal network and to the companion one-port passive distributed network. Hereafter, in order to relate the structure function to the spatial distributions of thermal properties $k(\mathbf{r})$ and $c(\mathbf{r})$, the companion wave propagation problem and the companion one-port passive distributed network are considered.

## 4 Procedure for Determining Structure Functions

The lumped LC ladder in Fig. 2 can be determined by applying the continuous fraction expansion of the $\mathsf{Z}_{LC}(s)$ impedance. Precisely for $k = 1, 2, \ldots$

$$C_{2k-1} = \lim_{s\to\infty} 1/s\mathsf{Z}_{2k-2}(s)$$
$$L_{2k} = \lim_{s\to\infty} \mathsf{Z}_{2k-1}(s)/s$$

in which $\mathsf{Z}_0(s) = \mathsf{Z}_{LC}(s)$ and

$$\mathsf{Z}_{2k-1}(s) = 1/(1/\mathsf{Z}_{2k-2}(s) - sC_{2k-1}),$$
$$\mathsf{Z}_{2k}(s) = \mathsf{Z}_{2k-1}(s) - sL_{2k}.$$

The continuous fraction expansion breaks down after $p$ lumped elements are determined. It can be easily proved that $p$ is the maximum order of the derivatives of $z_{LC}(t)$ with respect to time existing at $t = 0$. Equivalently $p$ can be related to the spatial regularity of the $k(\mathbf{r})$, $c(\mathbf{r})$ and $g(\mathbf{r})$ functions.

The determination of the non-singular LC transmission line in Fig. 2 can be obtained from $\mathsf{Z}_p(s)$ as follows. It results

$$A(0) = 1/\lim_{s\to\infty} \mathsf{Z}_p(s).$$

The other values of $A(\tau)$ can be determined by generalizing [14]. The impulse response $z_p(t)$ is computed. Then the following Fredholm integral equation is solved for $I(\tau, t)$

$$I(\tau, t) + \frac{A(0)}{2} \int_{-\tau}^{\tau} z_p(|t - r|) I(\tau, r)\, dr = 1$$

in which

$$-\tau \leq t \leq \tau.$$

It results in

$$\int_0^\tau A(\tau)\, d\tau = \int_0^\tau I(\tau, t)\, dt.$$

By recalling Eqs. (7)-(9), also $R$, $K(R)$ and $C(R)$ can be recovered from $A(\tau)$. This computation is stopped at $\tau = \tau_0$ such that $R(\tau_0) = R_0$. A procedure for determining structure functions is thus established.

It has to be noted that if all derivatives of $z_{LC}(t)$ exist at $t = 0$, then the continuous fraction expansion does not break down. In this case only an infinite lumped LC network is determined.

## 5 Analytical Example: Part I

A slab of thickness $L$, area $S$, thermal conductivity $k$ and volumetric heat capacity $c$ is considered. The slab extends in the region $\Omega$ in which $0 \leq x \leq L$, the power $P(t)$ is uniformly generated within the $\Sigma$ sub-region of $\Omega$, $0 \leq x \leq l \leq L$. On the surface $x = L$ of the boundary $\partial\Omega$ the temperature is set to the ambient temperature. On the rest of the boundary $\partial\Omega$ the thermal flux is set to zero. According to Eq. (6), the mean temperature rise in $\Sigma$ is the $T(t)$ temperature rise of a one-port passive distributed thermal network. By means of the procedure outlined in section 4, the structure function representation of the one-port passive distributed thermal network can be derived. The results for some values of $l$ are hereafter reported.

From the continuous fraction division one lumped capacitor is derived having capacitance

$$C_1 = lSc.$$

By solving the Fredholm integral equation the travel time representation follows from which the structure function representation is obtained. For $l = L$ it results in

$$\gamma = \frac{1}{(1 - 3\rho)^{\frac{1}{3}}}, \quad 0 < \rho < \frac{1}{3}.$$

For $l = 2L/3$ it results in

$$\gamma = \begin{cases} \dfrac{\frac{2}{3}}{\left(1 - \frac{9}{8}\rho\right)^{\frac{1}{3}}}, & 0 \leq \rho \leq \frac{37}{72} \\ \dfrac{\frac{4}{9}}{\left(\frac{5}{3} - 3\rho\right)^{\frac{1}{3}}}, & \frac{37}{72} \leq \rho \leq \frac{5}{9} \end{cases}$$

For $l = L/3$ it results in

$$\gamma = \begin{cases} \dfrac{\frac{2}{9}}{\left(\frac{8}{27} - \frac{2}{3}\rho\right)^{\frac{1}{3}}}, & 0 \le \rho \le \frac{7}{18} \\ \dfrac{\frac{2}{3}}{\left(\frac{16}{9} - 2\rho\right)^{\frac{1}{3}}}, & \frac{7}{18} \le \rho \le \frac{20}{27} \\ \dfrac{\frac{1}{3}}{\left(\frac{7}{9} - \rho\right)^{\frac{1}{3}}}, & \frac{20}{27} \le \rho \le \frac{7}{9} \end{cases}$$

For $l = L/6$ it results in

$$\gamma = \begin{cases} \dfrac{\frac{1}{18}}{\left(\frac{1}{27} - \frac{1}{6}\rho\right)^{\frac{1}{3}}}, & 0 \le \rho \le \frac{7}{36} \\ \dfrac{\frac{1}{6}}{\left(\frac{2}{9} - \frac{1}{2}\rho\right)^{\frac{1}{3}}}, & \frac{7}{36} \le \rho \le \frac{10}{27} \\ \dfrac{\frac{1}{3}}{\left(\frac{2}{3} - \rho\right)^{\frac{1}{3}}}, & \frac{10}{27} \le \rho \le \frac{13}{24} \\ \dfrac{\frac{5}{9}}{\left(\frac{40}{27} - \frac{5}{3}\rho\right)^{\frac{1}{3}}}, & \frac{13}{24} \le \rho \le \frac{32}{45} \\ \dfrac{\frac{5}{6}}{\left(\frac{25}{9} - \frac{5}{2}\rho\right)^{\frac{1}{3}}}, & \frac{32}{45} \le \rho \le \frac{95}{108} \\ \dfrac{\frac{1}{6}}{\left(\frac{4}{9} - \frac{1}{2}\rho\right)^{\frac{1}{3}}}, & \frac{95}{108} \le \rho \le \frac{8}{9} \end{cases}$$

Both cumulative and differential structure functions are shown in Figs. 3, 4 in which $\gamma = C/LSc$, $\rho = R/(L/kS)$, $\kappa = K/kcS^2$. As expected, as $l/L$ approaches 0 the structure function approaches that of a uniform RC transmission line.

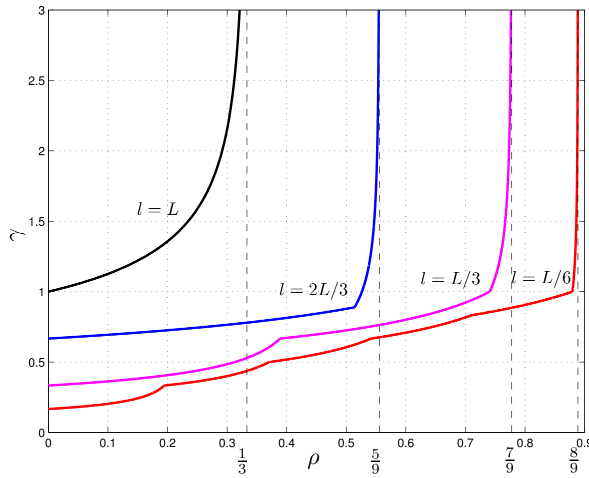


Figure 3: Cumulative Structure Functions.

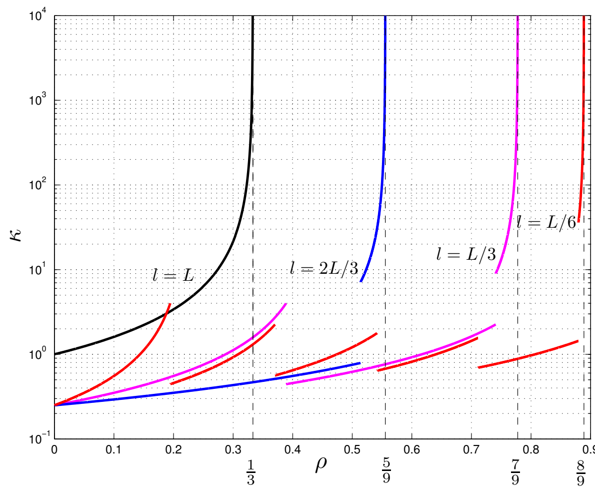


Figure 4: Differential Structure Functions.

## 6 Relation between Spatial Distributions of Thermal Properties and Structure Functions

Let $I(t)$ be a unit impulse and let $v(\mathbf{r}, t)$, $\mathbf{j}(\mathbf{r}, t)$ the solution of Eqs. (10)-(13).

Let $\omega_k(t)$ be the sub-region of $\Omega$ in each $\mathbf{r}$ of which $v(\mathbf{r}, r) \neq 0$ at some $r \le t$ and let $\partial\omega_k(t)$ be its boundary. Similarly let $\omega_k(t)$ be the sub-region of $\Omega$ in each $\mathbf{r}$ of which $\mathbf{j}(\mathbf{r}, r) \neq \mathbf{0}$ at some $r \le t$ and let $\partial\omega_k(t)$ be its boundary. It results $\omega_c(t) \supset \omega_k(t)$ and $\partial\omega_k(t) \supset \partial\omega_c(t)$. From Eqs. (10)-(13) it follows

$$v(\mathbf{r}, 0) = \frac{g(\mathbf{r})}{c(\mathbf{r})}.$$

Thus $\omega_c(0)$ is the support of $g(\mathbf{r})$, that is the $\Sigma$ region, while $\omega_k(0)$ is the sub-region of $\Sigma$ support of $\mathbf{j}(\mathbf{r}, 0)$. The $\partial\omega_c(t)$ and $\partial\omega_k(t)$ surfaces shown in Fig. 5 are the *wave fronts* of $v(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$ respectively, propagating at finite velocity given at each $\mathbf{r}$ by

$$\sqrt{\frac{k(\mathbf{r})}{c(\mathbf{r})}}.$$

Thus $\partial\omega_c(t)$ has equation $t = \psi_c(\mathbf{r})$ in which $\psi_c(\mathbf{r})$ is the solution of the *eikonal* equation

$$(\nabla\psi_c(\mathbf{r}))^2 = \frac{c(\mathbf{r})}{k(\mathbf{r})} \tag{18}$$

such that $\psi_c(\mathbf{r}) = 0$ defines the surface $\partial\omega_c(0)$. Similarly $\partial\omega_k(t)$ has equation $t = \psi_k(\mathbf{r})$ in which

$\psi_k(\mathbf{r})$ is the solution of the eikonal equation

$$(\nabla \psi_k(\mathbf{r}))^2 = \frac{k(\mathbf{r})}{k(\mathbf{r})} \qquad (19)$$

such that $\psi_k(\mathbf{r}) = 0$ defines the surface $\partial \omega_k(0)$. The family of *rays* orthogonal both to the family of wave fronts $\partial\omega_c(t)$ and $\partial\omega_k(t)$ is also considered. It results in

**Proposition 1** *For a ray $\gamma(t)$ going either from $\partial\omega_c(0)$ to $\partial\omega_c(t)$ or from $\partial\omega_k(0)$ to $\partial\omega_k(t)$ it is*

$$t = \int_{\gamma(t)} \sqrt{\frac{c(\mathbf{r})}{k(\mathbf{r})}}\, d\gamma.$$

From the theory of characteristic lines [11], it follows that, $v(\mathbf{r}, t)$ is affected by all and only the values of $c(\mathbf{r})$ in $\omega_c(t)$ and of $k(\mathbf{r})$ in $\omega_k(t)$. Moreover, as a consequence of the particular definition of $V(t)$ given by Eq. (15), the following fundamental result can be proved

**Proposition 2** *$V(2t)$ is affected by all and only the values of $c(\mathbf{r})$ in $\omega_c(t)$ and of $k(\mathbf{r})$ in $\omega_k(t)$.*

Let $V(\tau, t)$ and $I(\tau, t)$ be solutions of Eqs. (16), (17) when $I(t)$ is a unit impulse. From section 4 it follows

**Proposition 3** *$V(2t)$ is affected by all and only the capacitances and inductances of the lumped LC ladder network and by the values of $A(\tau)$ in the interval $0 \le \tau \le t$.*

As a consequence of Propositions 1, 2 and 3, and of Eqs. (7)-(9), the following main result is derived

**Proposition 4** *For each $0 < R \le R_0$, the restriction of the structure function $K(\rho)$ to $0 \le \rho \le R$, is affected by all and only the values of $c(\mathbf{r})$ in $\omega_c(t)$ and of $k(\mathbf{r})$ in $\omega_k(t)$, being*

$$t = \int_{\gamma(t)} \sqrt{\frac{c(\mathbf{r})}{k(\mathbf{r})}}\, d\gamma = \int_0^R \sqrt{K(\rho)}\, d\rho \qquad (20)$$

*in which $\gamma(t)$ is any ray going either from $\partial\omega_c(0)$ to $\partial\omega_c(t)$ or from $\partial\omega_k(0)$ to $\partial\omega_k(t)$.*

Eq. (20) expresses the physical interpretation of structure functions for generic multi-directional heat-flows.

The natural manner for exploiting this result in applications is the following. Let $\mathcal{C}$ be an heat diffusion problem. Let $\mathcal{N}$ be a one-port passive distributed thermal network modeling $\mathcal{C}$ and let $K(R)$ be its structure function. Let $\partial\omega_c(t)$ and $\partial\omega_k(t)$ be the wave fronts of the companion wave propagation problem of $\mathcal{C}$, with their rays $\gamma(t)$ going from

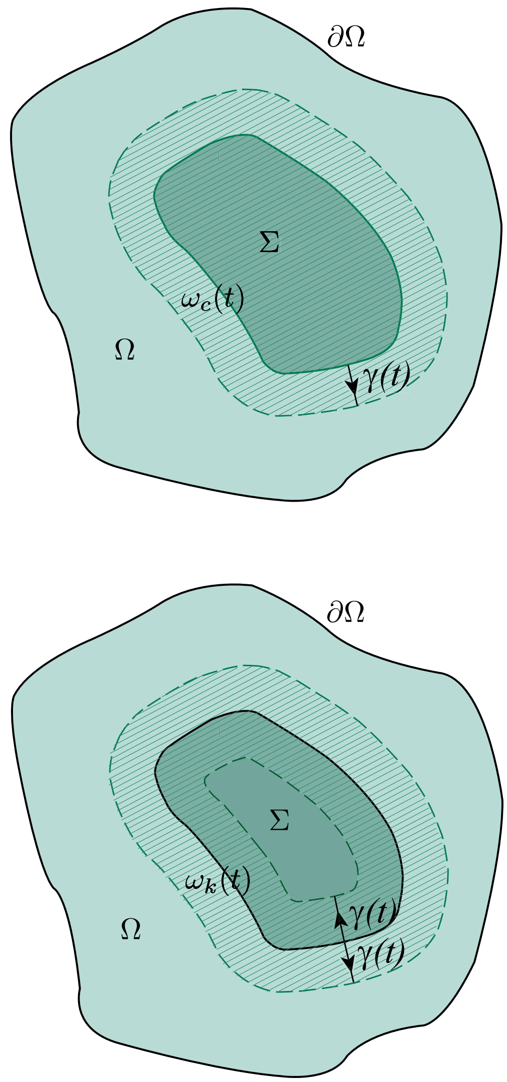


Figure 5: Propagation within $\Omega$ of the $\partial\omega_c(t)$ wave front of $v(\mathbf{r}, t)$ and of the $\partial\omega_k(t)$ wave front of $\mathbf{j}(\mathbf{r}, t)$.

$\partial\omega_c(0)$ to $\partial\omega_c(t)$ and from $\partial\omega_k(0)$ to $\partial\omega_k(t)$. Also let $\mathcal{C}'$ be a second heat diffusion problem whose thermal properties present a spatially localized difference with respect to $\mathcal{C}$. Let $\mathcal{N}'$ be the one-port passive distributed thermal network modeling $\mathcal{C}'$ and let $K'(R)$ be its structure function. If $R$ is the smallest value at which $K(R)$ and $K'(R)$ differ, either $\partial\omega_c(t)$ is the smallest surface at which the volumetric heat capacity of $\mathcal{C}$ and $\mathcal{C}'$ differ or $\partial\omega_k(t)$ is the smallest surface at which the thermal conductivity of $\mathcal{C}$ and $\mathcal{C}'$ differ, being

$$t = \int_{\gamma(t)} \sqrt{\frac{c(\mathbf{r})}{k(\mathbf{r})}}\, d\gamma = \int_0^R \sqrt{K(\rho)}\, d\rho = \int_0^R \sqrt{K'(\rho)}\, d\rho.$$

Moreover, by applying this procedure to different modified heat diffusion problems $\mathcal{C}'$, the quantity

$$\int_{\gamma(t_2)\setminus\gamma(t_1)} \sqrt{\frac{c(\mathbf{r})}{k(\mathbf{r})}}\, d\gamma$$

can be determined in which $\gamma(t_2)\setminus\gamma(t_1)$ is any ray going either from $\partial\omega_c(t_1)$ to $\partial\omega_c(t_2)$ or from $\partial\omega_k(t_1)$ to $\partial\omega_k(t_2)$.

## 7 Analytical Example: Part II

Let us consider the heat diffusion problem of section 5. The relation between the structure function $K(R)$ and the spatial distribution of thermal properties follows from Property 4. For each $R$ the restriction of the structure function to the interval $[0, R]$ is affected by all the values of the volumetric heat capacity in

$$\omega_c(t) = \left[0, \min\left(l + L\sqrt{\frac{k}{c}}\, t, L\right)\right]$$

and by all and only the values of thermal conductivity in

$$\omega_k(t) = \left[\max\left(0, l - L\sqrt{\frac{k}{c}}\, t\right), \min\left(l + L\sqrt{\frac{k}{c}}\, t, L\right)\right]$$

in which $t$ is determined by Eq. (20).

Let us now consider in details the case $l = L$, hereafter referred to as the heat diffusion problem $\mathcal{C}$. Such heat diffusion problem is transformed into the $\mathcal{C}'$ heat diffusion problem by doubling the thermal conductivity within $[0, 2L/3]$. By proceeding as in section 4, for the $\mathcal{C}'$ heat diffusion problem the following structure function is derived

$$\gamma = \begin{cases} \dfrac{1}{(1-3\rho)^{\frac{1}{3}}}, & 0 \le \rho \le \frac{19}{81} \\[2ex] \dfrac{\frac{3}{2}}{\left(\frac{23}{4} - \frac{81}{4}\rho\right)^{\frac{1}{3}}}, & \frac{19}{81} \le \rho \le \frac{23}{81}. \end{cases}$$

The differential structure functions of $\mathcal{C}$ and $\mathcal{C}'$ are compared in Fig. 6. The smallest value at which they differ is

$$R = \frac{19}{81}\frac{L}{kS}.$$

Thus from Proposition 4 it follows that $\mathcal{C}$ and $\mathcal{C}'$ present a difference in thermal conductivity at $\partial\omega_k(t)$ with

$$t = \int_{\gamma(t)} \sqrt{\frac{c(\mathbf{r})}{k(\mathbf{r})}}\, d\gamma = \frac{1}{3}\sqrt{\frac{c}{k}} L.$$

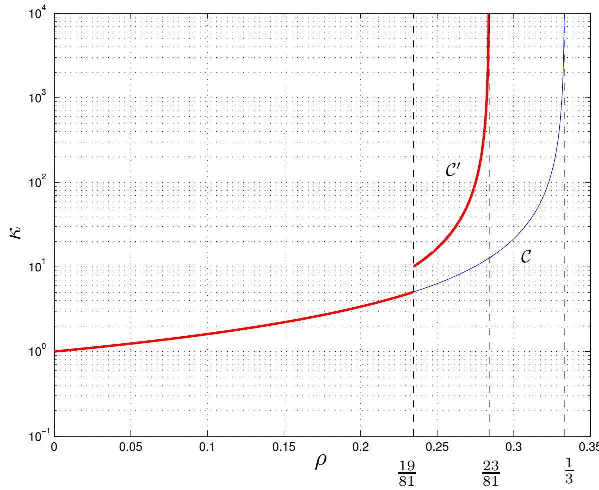


Figure 6: Differential Structure Functions of $\mathcal{C}$ and $\mathcal{C}'$.

## 8 Conclusions

A relation has been established between the structure function of the RC transmission line modeling a one-port passive distributed thermal network and the spatial distribution of thermal properties in heat diffusion problems with generic multi-directional heat-flows.

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