Amiet theory extension to predict leading-edge generated noise in compact airfoils

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Abstract
This paper extends the Amiet theory to frequencies where the airfoil can be considered a compact noise source. The original Amiet theory proposes to apply the Schwarzschild theorem in an iterative procedure, which generally leads to noise over-prediction at low-frequencies. To overcome this problem, this paper proposes two extra iterations to the theory aiming to improve convergence. Since the second iteration of the classical Amiet theory presents approximate analytical solutions, this paper proposes correction formulas for maintaining the solution accuracy of further iterations written in terms of analytical expressions. Results show that adding two extra iterations contribute to improved convergence, and consequently improved noise prediction, in the frequency range of interest for applications such as contra-rotating-open-rotors, wind-turbines and turbomachines. Comparison with experiments shows significant improvement at frequencies where the airfoil is considered a compact noise source.

Keywords: Amiet theory, airfoil noise, aeroacoustics

1. Introduction
The noise generated aerodynamically is a recurrent critical issue in applications of current large social interest, such as contra-rotating-open-rotors (CRORs) amongst other turbomachinery applications, high-density wind-farms and cooling fans found in air-conditioning, computers, and automotive applications. In those applications, the rotor blades are often subjected to an incoming turbulent inflow induced by upstream installation effects and aerodynamic elements. This turbulent and distorted inflow, when interacting with blade surfaces, causes pressure fluctuations and a consecutive noise generation. This noise source mechanism has been reported by numerous authors, through experimental [1, 2, 3, 4] and analytical analysis [5, 6, 7, 8, 9, 10].

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The development of better preliminary designs in the aforementioned applications, aimed at reducing noise emissions, is increasingly based on complex optimization procedures. However, at state-of-art, numerical aeroacoustic prediction approaches require excessively long turn-around time to make viable their implementation in efficient industrial routines. An alternative approach to this issue makes use of semi-analytical techniques, which are consistently raising as efficient and reliable tools for airfoil noise prediction [5, 9, 11].

Many of the current methodologies for analytical turbulence-airfoil noise predictions are based on Amiet’s theory [12]. This theory involves two major steps. Firstly, the airfoil aerodynamic response to a periodic monochromatic gust is calculated through a linearization of the problem, to obtain the surface forces. In a second step, Curle’s analogy is used to propagate the far-field noise [5]. Simple solutions can be obtained assuming parallel incoming gusts and infinite span airfoils, and this theory has been successfully applied to the prediction of airfoil-vortex noise cases [13] and helicopter blades noise prediction [7]. Roger and Moreau [14] extended the theory with a focus on skewed gusts and finite span airfoils, for the similar trailing-edge noise production problem. Their solution provides the airfoil response to supercritical and subcritical aerodynamic gusts, which have, respectively, velocity traces at the airfoil trailing-edge subsonic and supersonic. Roger and Moreau extension to the Amiet theory covered the airfoil self-noise mechanism, where the major noise source is localized at the trailing-edge, and following those works, Rozenberg [15] and Christophe [16] applied the technique to the leading-edge case.

An important aspect to include in the model is the mutual scattering of the acoustic perturbations, emitted predominantly at the leading-edge for the case of turbulence-interaction at high frequencies, by the trailing-edge. To address this mechanism, Roger and Moreau apply an iterative scattering procedure making use of Schwarzschild ’s theorem, and argue that two iterations of reciprocal back-scattering calculations are sufficient for convergence. While this is generally verified at high frequencies, i.e. non-compact airfoil chord, it will be shown below that in the compact regime the noise emissions can be significantly over-predicted, unless further iterations are carried out.

As a solution to this problem, this paper proposes the application of two additional Schwarzschild iterations, to improve the methodology convergence at small frequencies. Roger and Moreau’s extension to the leading-edge problem had only the first iteration calculated exactly, and high-frequency approximations were used to the second iteration computation. The derivations described below are performed for the second third and fourth iterations, essentially analytically, but with numerical treatment where necessary in order to remove such simplifications that would otherwise alter the accuracy.

To verify the theory developed in this paper, the predicted noise is compared with a turbulence-airfoil interaction noise experiment. The experiment is based on a NACA-0012 airfoil subjected to a grid generated turbulent flow at 30 m/s and observer localized at 0.80 m from the airfoil leading-edge in a angle of 90°.
2. Methodology and problem statement

2.1. Boundary value problem

For completeness, this section restates some key points of Amiet’s theory extended by Roger and Moreau [14], introducing the notations used in the remaining of the paper.

The Amiet theory adopts a flow model presented by Adamczyk [17], based on the Linearized Euler Equations. Those equations are written in terms of an incident and an scattered field. The scattered field is written in terms of a flow potential given by:

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c_0^2} \left( \frac{\partial^2 \phi}{\partial t^2} + 2U \frac{\partial^2 \phi}{\partial t \partial x} + U^2 \frac{\partial^2 \phi}{\partial x^2} \right) = 0 \quad (1)
\]

Along the Amiet formalism [12], we consider an airfoil lying on the plane \( z = 0 \) with chord comprised between \( 0 \leq \frac{x_0}{b} \leq 2 \) and span equal to \( s = 2d \), subjected to a mean flow with velocity \( U \) (see Fig. 1). The monochromatic gust is written as \( w(x_0, t) = w_0 \exp(i(\omega t - kx_0)) \), and the three boundary conditions for this problem are:

\[
\begin{align*}
\phi(x_0, y_0, 0, t) &= 0 & x_0 &\leq 0 \\
\frac{\partial \phi}{\partial z}(x_0, y_0, 0, t) &= -w & 0 &< x_0 \leq 2b \\
\frac{D \phi}{D t}(x_0, y_0, 0, t) &= 0 & x_0 &> 2b
\end{align*}
\]

These boundary conditions represent the zero velocity potential upstream of the leading-edge, zero normal velocity at the body surface, and zero pressure jump at the airfoil trailing-edge (Kutta condition) and downstream, respectively.
2.2. Problem statement

There is no formal proof of convergence of the iterative Schwarzschild procedure described above, and it has been so far assumed that after the second iteration, the residual flow potential $\phi_3(\bar{x}, 0)$ is quite small compared with the potential $\phi_0(\bar{x}, 0)$ obtained at the first iteration. This assumption can be quantitatively assessed in Fig. 2, showing these two potentials for different Helmholtz numbers $k_c$. The results indicate that in the compact chord regime $k_c \ll 1$, the residual flow potential $\phi_3(\bar{x}, 0)$ can reach up to 20% the initial flow potential $\phi_0(\bar{x}, 0)$. This relation thus represents a significant correction, and the question that naturally arises is about the further corrections that would be brought through further iterations, to eventually reach numerical convergence. The corresponding derivations are developed in the next Sections.

3. Further iterations of the Schwarzschild procedure: third iteration

In this section, the third iteration is developed, which imposes the boundary condition of zero flow potential at the region upstream the airfoil leading-edge.

3.1. Residual disturbance potential upstream of the airfoil trailing-edge

The residual potential is computed following the expression presented by Roger and Moreau [14], which relates the flow residual potential with the aerodynamic pressure

$$\phi(x_0, 0)^{(3)} = \frac{-b}{\rho U} \int_{-\infty}^{x} P_2(\xi, 0)e^{-ik_c(x-\xi)} d\xi$$

(5)
The flow potential due to the trailing-edge scattered pressure $P_{2}(\bar{x}, 0)$ is given by:

$$\phi(x_0, 0) = \frac{b w_0 e^{-i\pi/4} e^{-i\bar{k}_z x}}{\sqrt{2\pi(k_z + \beta^2 \kappa)}} \int_{-\infty}^{\bar{x}} [1 - (1 + i)E^*(2\kappa(2 - \bar{x}))] e^{-i(x - \bar{x})^2}$$

which can be solved by integration by parts and change of variables, leading to:

$$\phi(x_0, 0) = \frac{i b w_0 e^{-i\pi/4} e^{-i\bar{k}_z x}}{\sqrt{2\pi(k_z + \beta^2 \kappa)(\kappa - \bar{k}_z^*)}} \left[ (1 + i)E^*(2\kappa(2 - \bar{x})) \right] e^{-i(\kappa - \bar{k}_z^*)x}$$

Using the relation between the complementary error function and the Fresnel integral:

$$(1 + i)E^*(x) - 1 = -\text{erfc}^*(1 - i)\sqrt{\frac{x}{2}} = -\text{erfc}^*(1 + i)\sqrt{\frac{x}{2}}$$

it is possible to define the flow potential in terms of the complementary error function as:

$$\phi(x_0, 0) = \frac{-i b w_0 e^{-i\pi/4}}{\sqrt{2\pi(k_z + \beta^2 \kappa)(\kappa - \bar{k}_z^*)}} \left\{ -\text{erfc}^*(1 + i)\sqrt{\kappa(2 - \bar{x})} e^{-i(x - \bar{x})^2} \right\}$$

Since Eq. 9 is calculated using a high-frequency approximation of the trailing-edge scattered pressure $P_2(\bar{x}, 0)$, it is necessary to derive a correction function to $\phi(x_0, 0)^{(3)}$, valid in the low-frequency range as well.

To obtain a more precise value of $\phi(x_0, 0)^{(3)}$, the pressure $P_2(\bar{x}, 0)$ is numerically integrated, following Eq. 5. This numerical approach is verified to be accurate within 0.01%. In this procedure the correction function depends only on $\kappa$ and is identified as:

$$\mathcal{F}(\kappa) = \left( 1 + \frac{7}{90\kappa} \right)^{-1/3}$$

This correction function, shown in Fig. 3, is compared in Fig. 3 with numerical results for flow Mach numbers of $M = 0.25$, $M = 0.5$ and $M = 0.75$.

From Fig. 3 it is verified that the correction factor depends only on $\kappa$ and tends to unity for high-frequencies. The corrected leading-edge residual flow potential will be denoted by the superscript * and $\phi^*(x_0, 0)^{(3)} = \mathcal{F}(\kappa)\phi(x_0, 0)^{(3)}$.

For further analytical treatment, Eq. 9 requires an approximation, where the complementary error function is expanded as a power series and the zero-th
Figure 3: Correction factor for $\phi(x_0,0)^{(3)}$.

order term is considered ([18] (p. 297)), which is verified to be exact for larger arguments:

$$erfc[(1 + i)x] \approx \frac{(1 - 1i)e^{-2ix^2}}{2\sqrt{\pi x}}$$

(11)

Applying the approximation it is possible to simplify Eq. 9:

$$\phi^*(x_0,0)^{(3)} \approx F(\kappa)\frac{-(1 + i)bw_0e^{-i\pi/4}e^{-4i\kappa}}{\pi\sqrt{2(k_x + \beta^2\kappa)(\kappa - k_x^2)}} \left( -\frac{e^{i(\kappa + \beta^2\kappa)^2x}}{2\sqrt{\kappa(2 - x)}} + \frac{\sqrt{\kappa}e^{2ik_x^*e^{i(\kappa + \beta^2\kappa)^2x}}}{\sqrt{\kappa + k_x^*}\sqrt{(2 - x)(\kappa + k_x^*)}} \right)$$

(12)

For the next iteration, the time- and space-Fourier transform for the leading-edge flow potential $\phi(x_0,0)^{(3)}$ are calculated:

$$\phi(x_0,0)^{(3)}exp(i\omega t) = \varphi^{(3)}(x_0,0)exp(i\gamma x_0)exp(i\omega t)$$

(13)

such that the Fourier transformed potential $\varphi^{(3)}(x_0,0)$ can be rewritten as:

$$\varphi^{(3)}(x_0,0)^* \approx F(\kappa)\frac{-(1 + i)bw_0e^{-i\pi/4}e^{-4i\kappa}}{\pi\sqrt{2(k_x + \beta^2\kappa)(\kappa - k_x^2)}} \frac{e^{i\kappa x}}{\sqrt{2 - x}}$$

(14)

3.2. Third iteration

The third iteration corrects the residual potential present in the region upstream the airfoil leading-edge imposing the boundary condition of zero flow potential at the region upstream the airfoil leading-edge. This boundary value problem is solved by the application of the Schwarzschild theorem resulting into the leading-edge scattered potential $\Psi_3(\bar{x},0)$:

$$\Psi_3(\bar{x},0) = \frac{1}{\pi} \int_0^\infty \sqrt{\frac{\bar{x}}{\xi}} e^{-i\kappa(\xi + \bar{x})} \varphi^{(3)}(-\xi,0)^* d\xi$$

(15)
and substituting the potential \( \varphi^{(3)}(x_0,0)^* \) into Eq. 15 we find:

\[
\Psi_3(x,0) = \mathcal{F}(\kappa) \left( \frac{1 + i}{\pi^2 \sqrt{2(k_x + \beta^2\kappa)(2\sqrt{\kappa(\kappa + k_x^*)})}} \right) \int_0^\infty \frac{x}{\xi} \frac{1}{\xi + \sqrt{2 + \xi}} e^{-2i\kappa \xi} \sqrt{2 + \xi} d\xi
\]

(16)

The integral term of Eq. 16 does not allow an analytical solution, and we make use of the following high-frequency approximation:

\[
\int_0^\infty \frac{x}{\xi} \frac{1}{\xi + \sqrt{2 + \xi}} e^{-2i\kappa \xi} \sqrt{2 + \xi} d\xi \approx \int_0^\infty \frac{x}{2\xi} e^{-2i\kappa \xi} \sqrt{\frac{\sqrt{2 + \xi}}{\xi + \sqrt{2 + \xi}}} d\xi
\]

(17)

which after some algebra results in:

\[
\Psi_3(x,0) = \mathcal{F}(\kappa) \left( \frac{1 + i}{2\pi \sqrt{(k_x + \beta^2\kappa)(2\sqrt{\kappa(\kappa + k_x^*)})}} \right) [1 - (1 + i)E^*(2\kappa x)]
\]

(18)

The integral term of Eq. 17 gives an approximation for \( \Psi_3(x,0)^* \) valid for the high-frequency regime, which needs to be corrected for the low-frequency regime. To find a correction function to \( \Psi_3(x,0)^* \), Eq. 16 is numerically solved, with numerical accuracy of 0.01% and the comparison between the numeric solution and the analytical expression given by Eq. 18 shows that the correction function which satisfactorily corrects the error present at the low frequency regime calculation is:

\[
\mathcal{I}(\kappa) = \left( 1 + \frac{1}{4\kappa^{13/9}} \right)^{-1/3}
\]

(19)

Figure 4 illustrates the corrections computed for flow Mach number \( M = 0.25 \), \( M = 0.5 \) and \( M = 0.75 \):

From Fig. 4 it is verified that the correction function only depends of \( \kappa \) and it tends to zero for small values of \( \kappa \), while it tends to the unit when \( \kappa \) is
3.3. Pressure jump calculation

Once the leading-edge corrected potential \( \Psi^* (x, 0) \) is computed it is possible to derive the resulting pressure jump trace \( \mathbf{P}_3 (x_0, 0) \):

\[
\mathbf{P}_3 (x_0, 0) = P_3 (x_0, 0) e^{i \pi x_0} = - \frac{aU}{b} \left( \frac{\partial}{\partial \bar{x}} \phi (x_0, 0) + i k_x \phi (x_0, 0) \right) \]

where the derivative of the potential \( \phi^{(4)} (\bar{x}, 0) \) is expressed as:

\[
\frac{\partial \phi^{(4)} (\bar{x}, 0)}{\partial \bar{x}} = \frac{\partial \phi^{(3)*} (\bar{x}, 0)}{\partial \bar{x}} + \left( i k_x^* M^2 \psi_3^* (\bar{x}, 0) + \frac{\partial \psi_3^* (\bar{x}, 0)}{\partial \bar{x}} \right) e^{i k_x^* M^2 \bar{x}}
\]

The calculation of the term between parentheses yields:

\[
\left( i k_x^* M^2 \psi_3^* (\bar{x}, 0) + \frac{\partial \psi_3^* (\bar{x}, 0)}{\partial \bar{x}} \right) e^{i k_x^* M^2 \bar{x}} = \mathcal{F} (\kappa) \mathcal{I} (\kappa) f_1 \left\{ i (\kappa + k_x^* M^2) \left[ 1 - (1 + i) E^* (2 \kappa \bar{x}) \right] + (1 + i) \sqrt{\frac{\kappa^* e^{-2i \kappa \bar{x}}}{\bar{x}}} \right\} e^{i (\kappa + k_x^* M^2) \bar{x}}
\]

with:

\[
f_1 = \frac{(1 + i) b w_0 e^{(-\pi i/4)} e^{-i \kappa}}{4 \pi \sqrt{\kappa_x + \beta^2 \sqrt{\kappa^2 + k_x^* M^2}}} \]

and the derivative of the potential \( \phi^{(3)*} (\bar{x}, 0) \) on \( \bar{x} \) becomes:

\[
\frac{\partial \phi^{(3)*} (\bar{x}, 0)}{\partial \bar{x}} = \mathcal{F} (\kappa) f_2 (\alpha_1 + f_3 \alpha_2)
\]

with

\[
\alpha_1 = \left( -i (\kappa - k_x^* M^2) \left[ 1 - (1 + i) E^* (2 \kappa (2 - \bar{x})) \right] + (1 + i) \sqrt{\kappa e^{-4i \kappa \bar{x}} e^{2i \kappa \bar{x}}} \right) e^{i (\kappa - k_x^* M^2) \bar{x}}
\]

\[
\alpha_2 = \left( \frac{\sqrt{\kappa + k_x^* M^2}}{\sqrt{2 \pi}} (1 + i) e^{-2i (\kappa + k_x^*) \bar{x}} e^{i (\kappa + k_x^*) \bar{x}} - i k_x \left[ (1 + i) E^* (\kappa + k_x^*) \left[ 2 - \bar{x} \right] (\kappa + k_x^*) \right] \right) e^{-i k_x \bar{x}}
\]
and the constants given by

\[ f_2 = \frac{ib_0 e^{-i\pi/4}}{\sqrt{2\pi(k_x + \beta^2\kappa)(\kappa - k_x^*)}} \] (29)

\[ f_3 = -\frac{\sqrt{2\pi e^{-2i(\kappa - k_x^*)}}}{\sqrt{\kappa + k_x^*}} \] (30)

To facilitate the analytical calculations, the leading-edge scattered pressure \( P_3(x_0, 0) \) can be decomposed as a sum of 5 terms:

\[ P_3(x_0, 0) = \frac{-\rho U}{b} (\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5) \] (31)

where

\[ \beta_1 = -\mathcal{F}(\kappa)\mathcal{I}(\kappa)k_1(1 + i)\sqrt{\frac{\kappa}{\pi}} e^{-i(k - k_x^*)x} \] (32)

\[ \beta_2 = F(\kappa)\mathcal{I}(\kappa)ik_1(\kappa + k_x^* M^2 - \tilde{k}_x) (1 - (1 + i)E^* (2\kappa x)) e^{i(k - k_x^*)x} \] (33)

\[ \beta_3 = -F(\kappa)ik_3(\kappa - k_x^* M^2 - \tilde{k}_x) (1 - (1 + i)E^* [2\kappa (2 - \tilde{x})]) e^{-i(k - k_x^*)x} \] (34)

\[ \beta_4 = F(\kappa)k_3(1 + i) \left( \sqrt{\frac{\kappa}{\pi}} e^{-4i\kappa + k_1\sqrt{\kappa + k_x^*}} e^{-2i(k + k_x^*)} \right) \sqrt{\frac{2\pi}{\sqrt{\kappa + k_x^*}}} e^{i(k - k_x^*)x} \] (35)

\[ \beta_5 = F(\kappa) \left( -i\tilde{k}_x k_3 k_4 - k_2\sqrt{2\kappa} e^{-2i(\kappa - k_x^*)} \right) (1 - (1 + i)E^* (2 - \tilde{x})(\kappa + k_x^*)) e^{-i k_x^* \tilde{x}} \] (36)

and the constants are given by

\[ k_1 = \frac{(1 + i)b_0 e^{-i\pi/4} e^{-i\kappa}}{4\pi \sqrt{k_x + \beta^2\kappa(\sqrt{\kappa + k_x^*})}} \] (37)

\[ k_2 = -\frac{k_x b_0 e^{-i\pi/4}}{\sqrt{2\pi(k_x + \beta^2\kappa)(\kappa - k_x^*)}} \] (38)

\[ k_3 = \frac{ib_0 e^{-i\pi/4}}{\sqrt{2\pi(k_x + \beta^2\kappa)(\kappa - k_x^*)}} \] (39)

\[ k_4 = -\frac{\sqrt{2\pi e^{-2i(\kappa - k_x^*)}}}{\sqrt{\kappa + k_x^*}} \] (40)

It is verified that:

\[ \beta_4 = 0 \quad \beta_5 = 0 \] (41)

and, consequently, only the constants \( k_1 \) and \( k_3 \) are used.
4. Further iterations of the Schwarzschild procedure: fourth iteration

At this stage, the airfoil surface pressure and the corresponding aeroacoustic transfer function are computed starting from the third iteration. The Kutta condition and the boundary condition of zero pressure jump downstream the airfoil trailing-edge have now to be enforced.

4.1. Residual potential downstream of the airfoil trailing-edge

In the previous step the residual potential \( \phi(\bar{x}, 0)^{(3)} \) was computed for the region with \( \bar{x} \leq 2 \). Now, following the same approach, the potential \( \phi(\bar{x}, 0)^{(3)} \) is computed for the region with \( \bar{x} \geq 2 \):

\[
\phi(\bar{x}, 0)^{(3)} = -\frac{b}{\rho U} \int_{2}^{\bar{x}} 0 e^{-ik_x(2-\xi)}d\xi - \frac{b}{\rho U} \int_{-\infty}^{2} P_2(\xi, 0)e^{-ik_x(\bar{x}-\xi)}d\xi \quad \bar{x} \geq 2
\]

As the residual pressure at the region \( \bar{x} \geq 2 \) has been set to zero at the second iteration of the Amiet theory, we find:

\[
\phi(\bar{x}, 0)^{(3)} = \phi(2, 0)^{(3)} \quad \bar{x} \geq 2
\]

The total potential \( \phi^{(4)}(\bar{x}, 0) \) is therefore given as the superposition of the scattered potential \( \Psi_3(\bar{x}, 0) \) and \( \phi(2, 0)^{(3)} \):

\[
\phi^{(4)}(\bar{x}, 0) = \phi(2, 0)^{(3)} + \Psi_3(\bar{x}, 0) e^{i\bar{k}_x M^2 \bar{x}}
\]

which has Fourier components given by:

\[
\phi^{(4)}(\bar{x}, 0) = \phi(2, 0)^{(3)} + \Psi_3(\bar{x}, 0)
\]

The total potential \( \phi^{(4)}(\bar{x}, 0) \) implies the existence of a residual pressure \( P_3(\bar{x}, 0) \), defined for \( \bar{x} \geq 2 \), computed as:

\[
P_3(\bar{x}, 0) = \frac{\rho U w_0 F(\kappa) I(\kappa)(1 + i)e^{-i\pi/4} e^{-4i\kappa}}{4\pi \sqrt{k_x + \beta^2 k_x \sqrt{\kappa + \bar{k}_x}}} \left( i(\kappa + \bar{k}_x M^2 + \bar{k}_x)(1 - (1 + i)E^*(2\kappa \bar{x}))-\sqrt{\frac{\kappa}{\pi}}(1 + i)\frac{e^{-2i\kappa \bar{x}}}{\sqrt{\bar{x}}} \right) e^{i(\kappa + \bar{k}_x M^2)\bar{x}} + i \rho U b \Psi_3(\bar{x}, 0)^{(3)}
\]

This residual pressure \( P_3(\bar{x}, 0) \), defined for \( \bar{x} \geq 2 \), does not satisfy the boundary condition of zero pressure jump at the region downstream the airfoil trailing-edge, this boundary condition imposed, together with the non-penetration boundary condition for the region upstream the airfoil leading-edge conducts to a boundary value problem which can be solved by a fourth application of the Schwarzschild theorem.
4.2. Fourth iteration

The application of the Schwarzschild theorem allows the trailing-edge scattered pressure $P_4(\bar{x}, 0)$ computation as:

$$P_4(\bar{x}, 0) = -\frac{1}{\pi} \int_0^\infty G(\bar{x} - 2, \xi, 0) P_3(2 + \xi, 0) d\xi$$

$$P_4(\bar{x}, 0) = -\frac{1}{\pi} \int_0^\infty \sqrt{\frac{2 - \bar{x}}{\xi}} e^{-i\kappa(\xi + 2 - \bar{x})} P_3(\xi + 2, 0) d\xi$$  (47)

As done previously, we split the integral in Eq. 47 into different components with constants defined as:

$$k_1 = -\frac{\rho U w_0 F(\kappa)(1 + i) e^{-i\pi/4} e^{-4i\kappa}}{4\pi \sqrt{k_x + \beta^2 \kappa \sqrt{\kappa + k_y^*}}}$$  (48)

$$k_2 = i(\kappa + k_y^* M^2 + k_x)$$  (49)

$$k_3 = -i \frac{\kappa}{\pi} (1 + i)$$  (50)

At this stage each of the three components will be computed separately, where in a first step the trailing-edge scattered pressure $P_4(\bar{x}, 0)_{1,2,3}$ is determined, followed by the calculation of the non-dimensional pressure $g_{1,2,3}(\bar{x}, k_x, k_y)$ and finally the aeroacoustic transfer function $L(x, \kappa, k_y)_{1,2,3}$.

**The first term.** of the trailing-edge back scattered pressure $P_4(\bar{x}, 0)_1$ is:

$$P_4(\bar{x}, 0)_1 = -\frac{1}{\pi} k_1 k_2 \int_0^\infty \sqrt{\frac{2 - \bar{x}}{\xi}} \frac{e^{-i\kappa(\xi + 2 - \bar{x})}}{\xi + 2 - \bar{x}} (1 - (1 + i) E^* (2\kappa(\xi + 2))) e^{i\kappa(\xi + 2)} d\xi$$  (51)

The integrand has no primitive, so the large argument approximation to the Fresnel integral term is adopted:

$$(1 - (1 + i) E^* (2\kappa(\xi + 2))) \approx \text{erfc} \left( \frac{1 + i}{\sqrt{\kappa(2 + \xi)}} \right) \approx \frac{(1 - i) e^{-2i\kappa(2 + \xi)}}{2\sqrt{\pi} \sqrt{\kappa(2 + \xi)}}$$  (52)

Using this approximation, the term $P_4(\bar{x}, 0)_1$ can be computed as:

$$P_4(\bar{x}, 0)_1 \approx -\frac{1}{\pi} k_1 k_2 \frac{e^{-4i\kappa} e^{i\kappa(1 - i)}}{2\sqrt{\pi} \kappa} \int_0^\infty \sqrt{\frac{2 - \bar{x}}{\xi}} \frac{e^{-2i\kappa\xi}}{\xi + 2 - \bar{x}} d\xi$$  (53)

The integrand has still no analytical primitive, requiring a further high-frequency approximation:

$$\int_0^\infty \sqrt{\frac{2 - \bar{x}}{\xi}} \frac{e^{-2i\kappa\xi}}{\xi + 2 - \bar{x}} d\xi \approx \int_0^\infty \sqrt{\frac{2 - \bar{x}}{2\xi}} \frac{e^{-2i\kappa\xi}}{\xi + 2 - \bar{x}} d\xi$$  (54)
which has an analytical solution given by:
\[
\int_0^{\infty} \sqrt{\frac{2 - \bar{x}}{2\xi}} \frac{e^{-2i\xi\bar{x}}}{\xi + 2 - \bar{x}} d\xi = \frac{\pi e^{2i\kappa(2 - \bar{x})}}{\sqrt{2}} \left[1 - (1 + i)E^*(2\kappa(2 - \bar{x}))\right]
\] (55)

Replacing the integrand into Eq. 51 we obtain:
\[
P_4(\bar{x}, 0)_1 \approx \rho U w_0 \frac{F(\kappa)I(\kappa)(1 + i)e^{-4i\kappa}}{8\pi\kappa \sqrt{\pi(k_x + \beta^2\kappa)}} e^{-i\kappa\bar{x}} (1 - (1 + i)E^*(2\kappa(2 - \bar{x}))
\] (56)

After the calculation of the trailing-edge back scattered component \(P_4(\bar{x}, 0)_1\) the non-dimentionalized pressure \(g_4(\bar{x}, k_x, k_y)_1\) is given by:
\[
g_4(\bar{x}, k_x, k_y)_1 = \frac{F(\kappa)I(\kappa)(1 + i)e^{-4i\kappa}}{8\pi^2 \kappa \sqrt{\pi(k_x + \beta^2\kappa)}} e^{-i(\kappa - k_x^* M^2)(\xi + 1)} (1 - (1 + i)E^*(2\kappa(1 - \bar{x})))
\] (57)

and the aeroacoustics transfer function \(\mathcal{L}(x, k_x, k_y)_1\) is computed using:
\[
\int_{-1}^{1} e^{-i(\kappa - k_x^* M^2)(\xi + 1)} [(1 - (1 + i)E^*(2\kappa(1 - \xi))] e^{-i\mu(M - x/\sigma)} e^{-i\kappa(\xi + 2 - \bar{x})} d\xi = i \frac{e^{i\mu(M - x/\sigma)}}{(\kappa - \mu x/\sigma)}
\] (58)

resulting into:
\[
\mathcal{L}_4(x, k_x, k_y)_1 = -\frac{F(\kappa)I(\kappa)(1 + i)e^{-4i\kappa}}{8\pi^2 \kappa \sqrt{\pi(k_x + \beta^2\kappa)}} e^{i\mu(M - x/\sigma)} (1 - (1 + i)E^*(2\kappa(1 + \bar{x})))
\] (59)

The second term. of the trailing-edge scattered pressure \(P_4(\bar{x}, 0)_2\) is given by:
\[
P_4(\bar{x}, 0)_2 = -\frac{1}{\pi} k_1 k_3 \int_0^{\infty} \sqrt{\frac{2 - \bar{x}}{\xi + 2 - \bar{x}}} \frac{e^{-i\kappa(\xi + 2 - \bar{x})}}{\xi + 2 - \bar{x}} d\xi
\] (60)

which can be solved by using the approximation given by Eq. 54, with solution given by Eq. 55 resulting to:
\[
P_4(\bar{x}, 0)_2 \approx -k_1 k_3 \frac{e^{-i\kappa\bar{x}}}{\sqrt{2}} [1 - (1 + i)E^*(2\kappa(2 - \bar{x}))]
\] (61)

After some simplifications, we find:
\[
P_4(\bar{x}, 0)_2 \approx -\rho U w_0 \frac{F(\kappa)I(\kappa)(1 + i)e^{-4i\kappa}}{4\pi \sqrt{\pi(k_x + \beta^2\kappa)}(\kappa + k_x^*)} [1 - (1 + i)E^*(2\kappa(2 - \bar{x}))]
\] (62)
The non-dimensionalized pressure \( g_4(\bar{x}, k_x, k_y)_2 \) is given by:

\[
g_4(\bar{x}, k_x, k_y)_2 = \frac{\mathcal{F}(\kappa) I(\kappa)(1 + i)e^{-4i\kappa}e^{-i(\kappa - k_x^* M^2)(2\bar{x} - 1)}}{4\pi^2 \sqrt{\pi(k_x + \beta^2\kappa)(\kappa + k_x^*)}} [1 - (1 + i)E^*(2\kappa(1 - \bar{x}))]
\]

(63)

and the aeroacoustics transfer function \( \mathcal{L}(x, k_x, k_y)_2 \) is computed using the relation of Eq. 58:

\[
\mathcal{L}_4(x, k_x, k_y)_2 = \frac{\mathcal{F}(\kappa) I(\kappa)(1 - i)e^{-4i\kappa}e^{i\mu(M - x/\sigma)}}{4\pi^2 \sqrt{\pi(k_x + \beta^2\kappa)(\kappa + k_x^*)}} \left\{ e^{-2i(\kappa - \mu x/\sigma)} \left[ 1 - \sqrt{\frac{2\kappa}{\kappa + \mu x/\sigma}} (1 + i)E^*(2(\kappa + \mu x/\sigma))] - [1 - (1 + i)E^*(4\kappa)] \right] \right\}
\]

(64)

5. Results

5.1. Aeroacoustics transfer function comparison

To evaluate the methodology developed in this paper, the aeroacoustic transfer function \( \mathcal{L}(x, k_x, k_y)_2 \) is compared for different frequencies. In this evaluation,
the aeroacoustic transfer function is computed using the geometrical and flow parameters representative of the experiment subject of this paper. The airfoil chord is considered $c=0.1\, \text{m}$, the flow velocity is set to $30\, \text{m/s}$, the observer is localized at $0.8\, \text{m}$, measured from the airfoil leading-edge along a line perpendicular to the airfoil planform, and the frequency range selected for this study is representative of those of interest to this problem. Figure 5, Fig. 6 and Fig. 7 show the comparison.

![Figure 5](image1.png)  
(a) $k_c = 0.1; \ f = 54.11\, \text{Hz}; \ \kappa = 0.05$

![Figure 6](image2.png)  
(b) $k_c = 0.5; \ f = 270.56\, \text{Hz}; \ \kappa = 0.25$

Figure 5: Aeroacoustic transfer function.

From Fig. 5 is can be seen that for low-frequencies ($k_c = 0.1$ and $k_c = 0.5$), more iterations leads to a reduction to the predicted aeroacoustic transfer function. For $k_c = 0.5$, 4 iterations lead to an approximately converged noise polar, while for $k_c = 0.1$ further iterations are necessary to bring convergence.

![Figure 5](image3.png)  
(a) $k_c = 1; \ f = 541.13\, \text{Hz}; \ \kappa = 0.50$

![Figure 5](image4.png)  
(b) $k_c = 5; \ f = 2.70\, \text{kHz}; \ \kappa = 2.52$

Figure 6: Aeroacoustic transfer function.

For intermediate-frequencies ($k_c = 1$ and $k_c = 5$), increasing the number of iterations leads to increased noise levels. At this frequency regime, the airfoil switches from the compact to the non-compact regime, giving rise to interferential side-lobes in the noise radiation directivity. Also, with more iterations a larger noise radiation is observed in the downstream direction.
For the high-frequency regime ($kc = 10$ and $kc = 20$), increasing the number of iterations has a smaller influence on the predicted aeroacoustic transfer functions. The main influence is observed for listener angles corresponding to important interferential effects, where a reduction of the total predicted aeroacoustics transfer function is seen.

5.2. Experimental validation

To validate the theory proposed in this paper a turbulence-airfoil interaction noise case is chosen. The experimental set-up, assembled at the 4 m by 3 m by 3 m anechoic room installed at the von Kármán Institute for Fluid Dynamics, where further details about the experimental set-up can be found at [19]. The flow facility consists of 200 mm by 150 mm rectangular jet with the airfoil supported by side plates. In this analysis a 200 mm span by 100 mm chord NACA 0012 airfoil is subjected to a grid generated turbulent flow. The chosen mean flow velocity is 30 m/s and the flow turbulence, characterized hot-wire anemometry, is considered isotropic and uniform along the cross-section of interest for this experiment. At this flow condition, the integral turbulence correlation length is $\lambda = 0.34c$ and the velocity fluctuation root-mean-square is 1.99 m/s. In this work the turbulence energy spectrum showed to fit the von Kármán interpolation formula [20] as shown in Fig. 8. According Mish [3], as the flow approximates the airfoil, the turbulence is distorted and becomes anisotropic. This phenomena can be quantified by the Rapid Distortion Theory (RDT) described by Batchelor [21] and applied for the turbulence-airfoil case by Mish [3]. Following this last work, to compute the turbulence distortion tensor, this paper considers that the leading-edge radius plays the major role on the turbulence distortion and, the distortion tensor is computed using the solution of a potential flow around a cylinder. By applying this theory it is possible to compute the distorted normal turbulence energy spectrum at the airfoil leading-edge position, which is shown in Fig. 8.

For the noise measurement a 1/2 inch Bruel and Kjaer microphone was placed at 0.8 m from the airfoil leading-edge at a position of 90°. The micro-
phone signal is sampled by 60 s and the noise power spectral density is estimated using the Welch algorithm. For the noise power spectral density estimation, the hanning window is chosen, the block size of $2^{10}$ samples and an overlap of 50% is adopted. These parameters leads to a spectral frequency resolution of 50 Hz.

The noise prediction, based on the airfoil response to an aerodynamic gust, described in this paper, is made according the theory described by Amiet [5]. For the frequency range where the current airfoil is considered a compact noise source the parameter $\Lambda = M k_x d$ is not large enough to allow us to consider this an infinite span airfoil. This feature conveniently permits the validation of the current theory capability on predicting two-dimensional aerodynamic gusts and finite span airfoils. In this approach, the airfoil noise is calculated according to the relation:

$$S_{pp}(x, \omega) = \left( \frac{\rho_0 k_z b}{\sigma_0} \right)^2 \pi U d \int_{-\infty}^{\infty} \frac{\sin^2 [(K_y - k_y)d]}{\pi d(K_y - k_y)} \Phi_{ww}(K_x, k_y) |\mathcal{L}(x, K_x, k_y)|^2 dk_y$$

(70)

A comparison between the experimental sound spectra and the prediction is shown in Fig. 9.

Figure 9 shows that the higher number of iterations indeed conducts to a better noise prediction at frequencies where the airfoil is considered a compact noise source.
6. Conclusions

The developments detailed in this work are based on Amiet’s original theory [22], valid for one-dimensional gust and infinite span airfoils, and implement for the leading-edge case the extension proposed by Roger and Moreau [14] for the trailing-edge case. This extension allow us to consider two-dimensional aerodynamic gusts and finite span airfoils. A current limitation of the Amiet theory, extended by Roger and Moreau, is the typical noise over-prediction present at wavenumbers where the airfoil is considered a compact noise source. This paper proposes a solution to this problem by the application of two extra iterations of a methodology, solving the gust-airfoil boundary value problem by the iterative application of the Schwarzschild theorem. Large arguments and/or high-frequency approximations are necessary to allow a completely analytical treatment. Since the major objective of this work is to analyze the mid- to low-frequency regime, correction functions are proposed on the basis of numerical integration, to alleviate the inaccuracies that would be otherwise implied by the approximations.

The results show that the resolution of two extra iterations changes the amplitude of the airfoil response independently of the frequency, with more pronounced effects in the compact regime. The predicted noise shows also a better convergence after increasing number of iterations. Verification against experiments shows that two more iterations remarkably reduces the noise over-prediction at frequencies where the airfoil is considered a compact noise source.
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References


