High Reynolds number $K - \epsilon$ model of turbulent pipe flows with standard wall laws: first quantitative results

M. GISSELBRECHT[a,b], E. PLAUT[a]

a. LEMTA, Université de Lorraine & CNRS, UMR 7563,
54500 Vandœuvre-lès-Nancy, France

b. Mines Nancy, Département Énergie : Production, Transformation
matthieu.gisselbrecht@mines-nancy.org
emmanuel.plaut@univ-lorraine.fr

Résumé :

On s'intéresse à l'application du modèle $K-\epsilon$ à haut nombre de Reynolds à des écoulements turbulents en tuyau. Le modèle $K-\epsilon$ ayant été conçu à l'origine pour des écoulements peu ou pas confinés, la question de sa validité pour un tel écoulement confiné se pose. On utilise pour conditions limites des « lois de paroi » du type (1). Le tuyau étant supposé long et le régime établi, les champs moyens, vitesse axiale U, énergie cinétique turbulente K et dissipation ϵ, ne dépendent que du rayon cylindrique r. Un système d'équations différentielles ordinaires (7 - 9) est obtenu, dans lequel le gradient de pression moyen G est aussi une inconnue. Ce système est discrétisé par méthode spectrale et collocation. Le système non linéaire obtenu est résolu par la méthode de Newton-Raphson. Les calculs ont été effectués pour un nombre de Reynolds Re défini en (2) variant de 2200 à 23200. Les résultats sont physiquement corrects, par exemple, les coefficients de frottement sont proches de la loi de Blasius. Ces résultats pourraient être utilisés pour valider des codes « industriels » . Des comportements asymptotiques sont mis en évidence quand $Re \to \infty$. Une amélioration du modèle est finalement proposée, pour mieux décrire les écoulements à nombre de Reynolds intermédiaire, $2000 \lesssim Re \lesssim 20000$.

Abstract :

We study turbulent pipe flow with the high-Reynolds number $K - \epsilon$ model, using wall laws as boundary conditions. The system (7 - 9) is discretised with a spectral method. We estimate all the spectral coefficients and the pressure gradient thanks to a Newton - Raphson method. The results are relevant physically, e.g., the friction factors found are close to the Blasius correlation. These results constitute a landmark that could be used to validate 'industrial' codes. An asymptotic regime is found at large Reynolds number. We also propose a slight possible optimization of the model to better describe intermediate Reynolds number cases.

Mots clefs : modèles de turbulence, conditions limites, lois de parois

1 Introduction

Turbulent flows are often encountered in industrial systems: around wind turbines or vehicles, in aerodynamic and combustion applications, in circuits of energy systems such as nuclear, thermal, hydraulic power stations, etc... In order to develop more efficient systems, one wants to optimize these flows. For this purpose, Computational Fluid Dynamics is often used. However, solving the Navier-Stokes equation directly, in 'Direct Numerical Simulations' (DNS), is difficult in turbulent flows, because the small spatial and temporal scales of the turbulence can hardly be resolved. This is especially true if complex geometries are dealt with.

Therefore, one often uses 'turbulence models', such as the $K - \epsilon$ model. It is well-known and often used in industry. With this model, the turbulent flow is characterized by 3 mean fields: the mean velocity $\overline{\mathbf{V}}$, the turbulence kinetic energy K and the dissipation rate ϵ. This model is valid in turbulent areas. It has been designed historically for free flows, i.e., plane jets and mixing layers (Launder & Spalding [1]).

For flows delimited by walls, where the viscous effects become important, the 'wall function' method has been developed to obtain boundary conditions on $\overline{\mathbf{V}}$, K and ϵ, as already advocated in [1], and described in Wilcox [2] or Davidson [3]. For instance, close to a smooth wall, one writes, in the streamwise direction,

$$V = \| \overline{\mathbf{V}} \| = u_\tau \left(\frac{1}{\kappa} \ln y^+ + C \right) , \tag{1}$$

with u_τ the friction velocity, $\kappa = 0.41$ the Von Karman constant, y the wall-normal coordinate, $y^+ = y u_\tau / \nu$, ν the kinematic fluid viscosity and C an additive constant. One problem with this wall law is that there does not exist a universal value of C relevant for all flows and Reynolds numbers. Even if the geometry of the flow is fixed, e.g. pipe flow, experiments and DNS show that $C \simeq 7$ for $Re \simeq 5500$, see Eggels *et al.* [4], figure 4a. However, $C \to 5.2$ for $Re \to \infty$, see El Khoury *et al.* [5], figure 6. Thus, C depends on Re. Importantly, the Reynolds number here

$$Re = 2Va/\nu , \tag{2}$$

with V the bulk velocity, a the pipe radius. Launder and Spalding [1] argued that for $Re < 20000$ one should not use the standard $K - \epsilon$ model with wall laws, or 'high Reynolds number model', but an alternative 'low Reynolds number model', with supplementary terms in the standard $K - \epsilon$ equations. This model computes the flow up to the wall of a pipe and has boundary conditions directly at the wall. Concerning the mean velocity and friction factor, this model was successful at moderate Reynolds number (see the figure 3.3 in [1]). This model is rather complex, though, and requires a very fine meshing near the wall. Therefore, calculation time and computational power are much larger than the ones required for a model using wall laws.

To our knowledge, no quantitative results exist in the literature concerning the standard high Reynolds number $K - \epsilon$ model of turbulent pipe flows, using wall laws. Indeed, the information given by Launder and Spalding [1] about this model are only qualitative. In order to fill this gap, we present here accurate spectral solutions of this model. The obtained results are encouraging and tend to prove that the standard $K - \epsilon$ model can be relevant for pipe flows.

2 Presentation of the model

2.1 Dimensionless parameters and equations

Consider the flow of a Newtonian, incompressible fluid in a straight pipe of radius a. The pipe is long and we disregard inlet or outlet effects. We use cylindrical coordinates $(r,\ \theta,\ z)$ with Oz the axis of the pipe. The length scale is a, the velocity scale is the mean value U_0 of the centerline velocity. The main control parameter is the Reynolds number

$$R \ = \ U_0\, a/\nu\ . \tag{3}$$

For symmetry reasons, the mean fields of the model are of the form

$$\overline{\mathbf{V}} \ = \ U(r)\overline{\mathbf{e}}_z \ \ , \ \ K \ = \ K(r) \ \ , \ \ \epsilon \ = \ \epsilon(r) \ \ . \tag{4}$$

The flow is pressure-driven: if $\hat{P} = P + \rho g Z$ with P the mean pressure, ρ the density, g the acceleration due to gravity, Z the vertical coordinate, one has

$$\hat{P} \ = \ \hat{P}_0 - Gz \quad \text{with} \quad G \quad \text{the mean driving pressure gradient.} \tag{5}$$

The turbulent kinematic viscosity

$$\nu_t \ = \ C_\nu K^2/\epsilon \quad \text{with} \quad C_\nu \ = \ 0.09\ . \tag{6}$$

The radial and wall-normal coordinates are normalized by a, the mean velocity by U_0, the turbulence kinetic energy by U_0^2, the dissipation rate by U_0^3/a, the turbulent viscosity by ν, the pressure gradient by the laminar pressure gradient

$$G_0 \ = \ 4\rho\nu U_0/a^2\ .$$

From now on all quantities are dimensionless. Denoting with a prime the derivative with respect to r, from the standard high Reynolds number $K-\epsilon$ equations, one obtains

$$\nu_t U'(r) + 2Gr \ = \ 0\ , \tag{7}$$

$$\frac{1}{r}\frac{d}{dr}\left[r\nu_t K'(r)\right] + \nu_t[U'(r)]^2 - R\epsilon(r) \ = \ 0\ , \tag{8}$$

$$\frac{1}{\sigma_\epsilon}\frac{1}{r}\frac{d}{dr}[r\nu_t\epsilon'(r)] + C_1\nu_t\frac{\epsilon(r)}{K(r)}\,[U'(r)]^2 - C_2 R\frac{[\epsilon(r)]^2}{K(r)} \ = \ 0\ , \tag{9}$$

$$\text{with} \quad \nu_t \ = \ C_\nu R K^2/\epsilon \quad , \quad \sigma_\epsilon \ = \ 1.3 \quad , \quad C_1 \ = \ 1.44 \quad , \quad C_2 \ = \ 1.92\ . \tag{10}$$

2.2 Boundary conditions - Wall laws

Since, in the system formed by (7 - 9), G is unknown, six boundary conditions are required. As the mean velocity is normalized by the centerline velocity,

$$U(0) \ = \ 1\ . \tag{11}$$

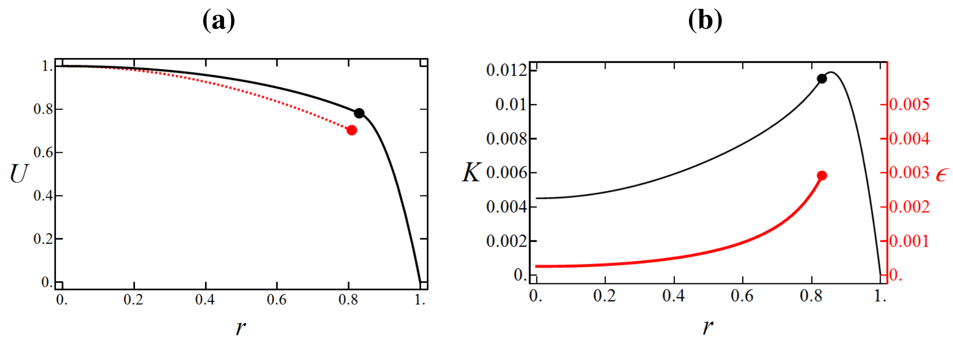

Figure **1** : Mean fields for $R = 3000$, as a function of the dimensionless radius r. The disks show the location of the radius r_0, given by (13), where the boundary conditions apply. **(a)**: Mean velocity $U(r)$ after the spectral computations (full line) and initial guess (dashed line). **(b)**: Turbulence kinetic energy $K(r)$ (thin line) and dissipation rate $\epsilon(r)$ (thick line) after the spectral computations.

The three following boundary conditions concern the area near the wall. In order to establish them, we use the wall functions, see e.g. Wilcox [2]. Let us denote r_0 a typical radial coordinate in the 'overlap' layer (or 'Log layer' according to Wilcox). To estimate it, one uses the nondimensional friction velocity, obtained from a balance of forces in a slice of fluid,

$$u_\tau = \sqrt{2G/R}\,. \tag{12}$$

The reduced coordinate y^+ included in the logarithm in (1) is a control parameter that one fixes, $y_0^+ = 30$. Thus the corresponding radius

$$r_0 = 1 - y = 1 - y_0^+/\sqrt{2RG}\,. \tag{13}$$

The dimensionless form of (1) is

$$U(r = r_0) = \sqrt{\frac{2G}{R}}\left(\frac{1}{\kappa}\ln y_0^+ + C\right). \tag{14}$$

Hereafter, we use $C = 5.2$ inspired by the results of [5]. Besides, Wilcox [2] writes that the turbulence kinetic energy only depends on the friction velocity. By using the expression (12), one finds

$$K(r = r_0) = \frac{u_\tau^2}{\sqrt{C_\nu}} = \frac{2}{\sqrt{C_\nu}}\frac{G}{R}\,. \tag{15}$$

Moreover, still following [2], one has

$$\epsilon(r = r_0) = \frac{u_\tau^3}{\kappa y} = \frac{1}{\kappa}\left(\frac{2G}{R}\right)^{\frac{3}{2}}\frac{1}{1 - r_0} = \frac{4G^2}{R\kappa y_0^+}\,. \tag{16}$$

Finally, the fields K and ϵ have to be regular at the center of the pipe,

$$K'(0) = 0 \quad , \quad \epsilon'(0) = 0. \tag{17}$$

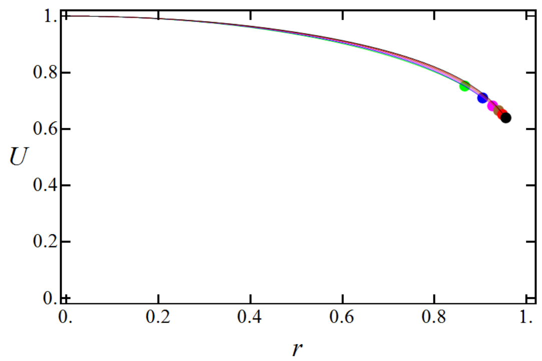

Figure **2** : Mean velocity $U(r)$ for $R = 4000$ to 14000 by steps of 2000. The disks indicate the radius r_0 given by (13), that increases when R increases.

3 Spectral method

We assume that all fields are regular, i.e. analytic. Therefore, as explained for instance in Priymak and Miyazaki [6], they can be written as power series of r^2. For numerical reasons, we rather use even Chebyshev polynomials, i.e. write

$$U(r) = \sum_{n=0}^{N} a_n T_{2n}\left(\frac{r}{r_0}\right), \tag{18}$$

with $N+1$ the number of modes, T_{2n} the $2n^{th}$ order Chebyshev's polynomial of the first kind, of degree $2n$. We write the turbulence kinetic energy K and the dissipation rate ϵ similarly. The corresponding polynomial coefficient are denoted b_n and c_n respectively. Hence, the vector

$$X = \left(a_0 \dots a_N \, , \, b_0 \dots b_N \, , \, c_0 \dots c_N \, , \, G \right)^T, \tag{19}$$

has to be computed. We choose the following collocation points,

$$r_i = \cos\left(\frac{(2i-1)\pi}{2(2N+1)}\right), \tag{20}$$

where i varies from 1 to N, thus $r_i \in]0\,,\,1[$. Then, we construct the function $F(X)$ that collects all equations to be solved numerically. It is a vector of length $3N+4$ in which there are equations (7 - 9) evaluated at the collocation points (20) on the one hand, and the boundary conditions (11), (14), (15), (16) on the other hand. Finally, the nonlinear system $F(X) = 0$ is solved with a Newton-Raphson method.

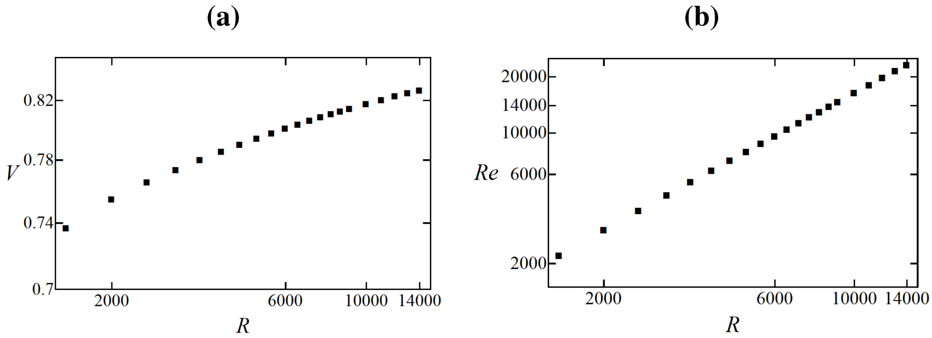

Figure 3 : Global values vs the centerline velocity Reynolds number R. **(a)**: Bulk velocity (log-log scale). **(b)**: Bulk velocity Reynolds number Re.

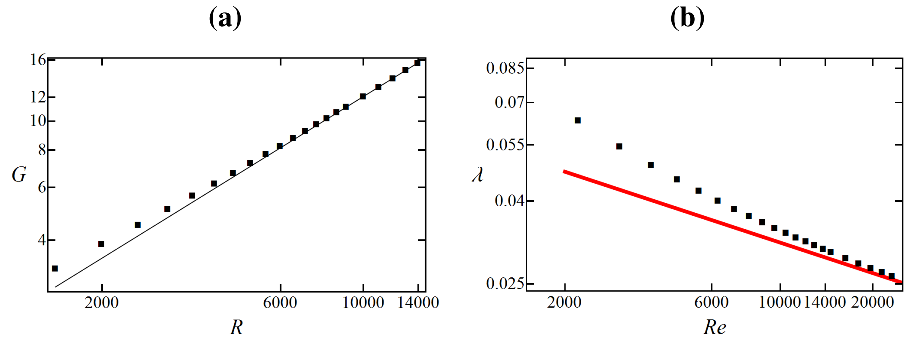

Figure 4 : **(a)**: Pressure gradient vs R (squares) and power law (24) (continuous line). **(b)**: Friction factor vs Re (squares) and Blasius correlation (22) (continuous line).

4 Results

4.1 Friction factor - Initial guess

We first estimate the pressure gradient and the fields at $R = 3000$ to obtain an initial value of X. Concerning the pressure gradient G, it is related to the friction factor

$$\lambda = 16G/(RV^2) . \tag{21}$$

This factor can be estimated from the Blasius correlation

$$\lambda_B = 0.316 Re^{-1/4} , \tag{22}$$

relevant for $Re \in [3000 , 100000]$ according to [7]. Regarding Re (2) and R (3), one has

$$Re = 2VR . \tag{23}$$

Assuming from [4] that $V = 0.75$, we obtain $G = 4.1$.

We approximate the profile of each fields U, K and ϵ by a polynomial of degree 2, increasing with r for K and ϵ, decreasing with r for U, inspired by the boundary conditions (11) and (14 - 16).

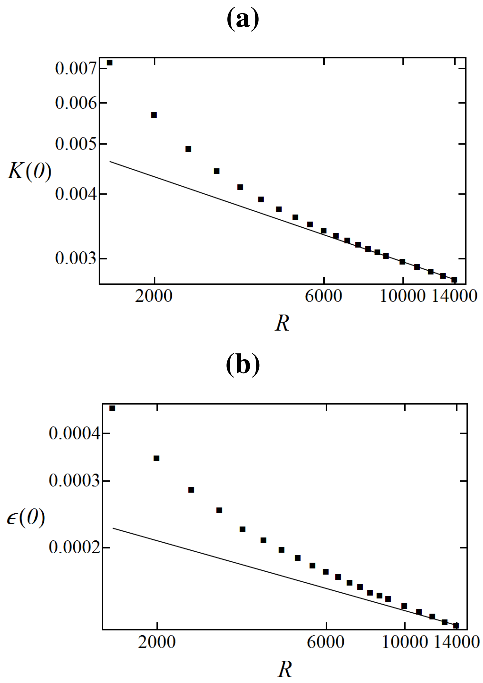

Figure **5** : Values of the turbulent fields at the center of the pipe, $r = 0$, vs R: the squares show the numerical results and the full lines the power laws (26). **(a)**: Kinetic energy. **(b)**: Dissipation rate.

4.2 Overview of the results

Starting at $R = 3000$, $N = 8$ with the initial guess explained hereabove, the Newton method converged to the solution displayed in figure 1, and to $G = 5.09$. Since G evolved, r_0 given by (13) evolved between the initial guess and the real solution, as visible in figure 1a. For $r \in [r_0, 1]$, we extend the function $U(r)$ by a polynomial of degree 2, that vanishes at $r = 1$ and insures that the complete function $U(r)$ is $C^1([0,1])$. This extension is shown in figure 1a. We do the same for the turbulence kinetic energy K, which also fulfills $K(r = 1) = 0$, to obtain the complete profile of figure 1b. Concerning the turbulence dissipation rate ϵ, its value as $r \to 1$ is unknown, therefore, we do not extend the profile of ϵ for $r > r_0$ (figure 1b).

The validity of the results is checked by computing the residuals of the eq. (7 - 9) and of the boundary conditions. To obtain a good convergence, $N = 8$ is sufficient at $R = 3000$, but $R = 14000$ needs $N = 23$. Starting from the solution at $R = 3000$, R has been decreased and increased by small steps, to obtain solutions in the range $1500 \leq R \leq 14000$. Interestingly, the mean velocity profiles seem to collapse onto a master curve, or asymptotic solution, as visible in figure 2. The corresponding bulk velocity, computed using the extended profiles as visible in figure 1a, is shown in figure 3a. We expect, from the observation made on figure 2, that $V \to V_\infty$, a finite value, as $R \to \infty$, but obviously this will only happen at very large Reynolds number. From equation (23), we obtain the bulk Reynolds number Re as shown in figure 3b.

The figure 4a shows that the mean pressure gradient follows a power law at high Reynolds number,

$$G \sim G_0 R^{\alpha_G} \quad \text{as} \quad R \to \infty \,. \tag{24}$$

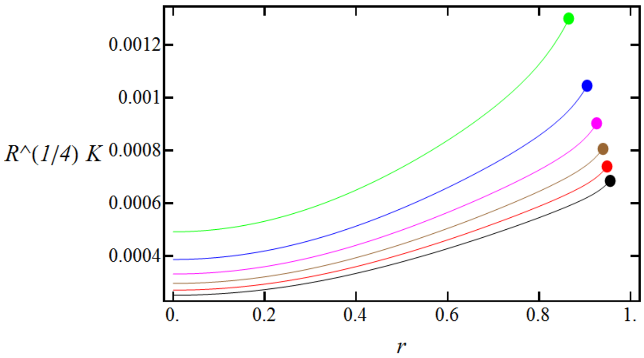

Figure **6** : Same as figure 2, but for the rescaled turbulence kinetic energy $R^{1/4}K(r)$.

This, together with the hypothesis $V \to V_\infty$, would yield, for the friction factor (21),

$$\lambda \sim 16G_0 R^{\alpha_G - 1}/V_\infty^2 \sim 16G_0(2V_\infty)^{1-\alpha_G} Re^{\alpha_G - 1}/V_\infty^2 \sim \lambda_0 Re^{\alpha_G - 1} , \tag{25}$$

which would agree with the Blasius correlation (22) provided that $\alpha_G = 3/4$. In fact, with $\alpha_G = 3/4$ and the requirement that $G(R = 14000) = G_0\ 14000^{3/4}$, we obtain $G_0 = 0.012$ and the full line in figure 4a, which describes all numerical results as soon as $R \gtrsim 6000$. Figure 4b shows that the friction factors of the $K - \epsilon$ model seem to approach asymptotically the ones given by the Blasius correlation, as $Re \to \infty$.

In order to characterize the fields K and ϵ as $R \to \infty$, we plot in figure 5 the values of $K(r = 0)$ and $\epsilon(r = 0)$ vs R. These centerline values follow, as $R \to \infty$, power laws:

$$K(r = 0) \sim 2.6 \cdot 10^{-3} R^{-0.25} \quad , \quad \epsilon(r = 0) \sim 4.4 \cdot 10^{-4} R^{-0.37} . \tag{26}$$

The coefficients and exponents have been obtained from fits using the two second-to-last points of the data $K(0)$ and $\epsilon(0)$ shown in figure 5. The power law on $K(0)$ (resp. $\epsilon(0)$) is a good approximation for $R \gtrsim 8000$ (resp. $R \gtrsim 10000$). If one inserts the power law (24) for G in the boundary condition (15) on K, one obtains

$$K(r = r_0) = 2G/(\sqrt{C_\nu} R) \sim 2G_0 R^{-1/4}/\sqrt{C_\nu} \quad \text{as} \quad R \to \infty . \tag{27}$$

Thus, both $K(r = 0)$ and $K(r = r_0)$ are proportional to $R^{-1/4}$ at large Reynolds number. This has motivated us to construct the figure 6 which displays $R^{-1/4}K$ vs r. We conjecture that $R^{1/4}K(r)$ should converge to an asymptotic universal profile as $R \to \infty$, but this should be studied with computations at higher Reynolds number. Concerning ϵ, note that there is a disproportion between $\epsilon(r = 0)$, proportional to $R^{-0.37}$ according to (26), and $\epsilon(r = r_0)$, which, according to (16) and (24), is equal to

$$\epsilon(r = r_0) = 4G^2/(R\kappa y_0^+) \sim 4G_0^2 R^{1/2}/(\kappa y_0^+) \quad \text{as} \quad R \to \infty . \tag{28}$$

Hence, clearly, no rescaled profile of ϵ can converge to an asymptotic universal profile.

C	V	Re	G	r_0	λ	λ/λ_B
5.2	0.755	3020	3.89	0.759	0.0546	1.28
6.0	0.756	3026	3.59	0.750	0.0501	1.18
7.0	0.757	3031	3.25	0.737	0.0453	1.06
7.6	0.758	3032	3.07	0.729	0.0428	1.00

Table 1: For $R = 2000$, features of solutions with various values of the additive constant C.

5 Concluding discussion

A systematic study of the high-Reynolds number $K - \epsilon$ model of turbulent pipe flow, using wall laws, has been performed. The ordinary differential equations of the model (7-9) have been discretised with a spectral method and collocation, and solved with a Newton method. The quantitative results obtained, displayed in figure 4b, confirm the qualitative statement of Launder & Spalding [1], p. 283: *'If these models are tuned to give correct predictions for $Re > 20000$, the friction factor at low Reynolds numbers is invariably predicted too high'*. It should be noted, however, that no 'tuning' of the $K - \epsilon$ model has been made here: we have used the standard $K - \epsilon$ coefficients recommended in [1] or [2], and the value of the additive constant in the wall law for the velocity (1) $C = 5.2$ [5]. With this in mind, the fact that the friction factors of the model seem to converge to the Blasius correlation as $Re \to \infty$ (figure 4b) is remarkable. Also, the simple 'universal form' of the velocity profile visible in figure 2 is interesting. This study should be extended to larger values of the Reynolds number, where it could be that the bulk velocity V converges to a finite value V_∞ (see figure 2 and 3a), and $R^{1/4}K$ converges to an asymptotic universal profile (see figure 6). For this purpose, the presently available code will have to be improved. It would also be interesting to compare the results of the model with the ones of DNS.

In order to better describe the flows at intermediate Reynolds number,

$$2000 \lesssim Re \lesssim 20000 ,$$

we propose to tune the additive constant C in the wall law for the velocity (1). The computational cost of such a modified model is the same as for the standard $K - \epsilon$ model. A preliminary test has been performed as follows: for $R = 2000$, $Re \simeq 3000$, which corresponds to the lowest Reynolds number for which the Blasius law is relevant [7], we have computed solutions with increasing values of C, as shown in table 1. The bulk velocity V is almost unchanged, but the mean pressure gradient G decreases significantly when C increases. Consequently, the friction factor λ decreases and approaches the value given by the Blasius law. We suggest the use of a modified $K - \epsilon$ model where C would be a smooth decreasing function of R, with, for instance, $C(R = 2000) = 7.6$ and $C(R = 14000) = 5.2$. A systematic test of this idea is planned.

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