Onset-of-instability in axially compressed rectangular and hexagonal honeycomb

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Abstract

We propose a theoretical approach to study instabilities in perfect honeycomb of infinite extent under general loading conditions based entirely on unit-cell calculations. It combines Bloch wave representation theorem and the analytical solution of the linearized von Kármán plate equation. Several geometries and loading cases are investigated. The results show that the buckling mode is highly dependent on the type of loading. The addition of transverse shear not only reduces the critical axial strain, but also affects the wavelengths of the critical eigenmode.

1 Introduction

One of the many uses of honeycomb is as core in sandwiched plates, producing very high stiffness-to-weight ratio structures. The macroscopically observed crushing mechanism of these structures has its origin in instabilities at the local scale. Of particular interest here are the critical (i.e. onset of a buckling-type instability) loads and corresponding eigenmodes of honeycomb under general 3D loading involving simultaneous axial compression and transverse shear.

Since the critical eigenmodes in honeycomb often involve more than one unit cell, numerical studies are limited by the size of the domain considered for their analyses. The theoretical method presented makes use of the Bloch wave representation theorem for the eigenmode, and so it is sufficient to solve the linearized von Kármán plate equations for the smallest representative volume element.

2 Formulation

2.1 Model description

Three different geometries are considered: rectangular honeycomb with a varying in-plane aspect ratio, an isotropic-section hexagonal honeycomb (all plates with equal thickness) and an anisotropic-section hexagonal honeycomb (one set of plates with double thickness, corresponding to the usual manufacturing procedure that uses strips of glue on initially flat plates that are subsequently pulled apart).

The proposed approach makes use of the fact that the honeycomb plates remain flat in the principal solution prior to the onset of the first instability, as shown experimentally [3] for axially compressed Al honeycomb, and solves analytically the corresponding eigenvalue problem.
each cell wall, von Kármán plate theory dictates that the partial differential equations for the out-of-plane \((w)\) and in-plane \((u_\alpha)\) eigenmode component, expressed in local coordinates, are:

\[
\begin{align*}
D \Delta^2 w - \sigma_{\alpha\beta} w_{,\alpha\beta} &= 0, \\
L_{\alpha\beta\gamma\delta} u_{,\gamma\delta} &= 0,
\end{align*}
\]

where \(D\) is the bending stiffness, \(t\) is the plate thickness, \(E\) its Young modulus, \(\nu\) its Poisson ratio, \(\sigma\) is the in-plane stress of the plate, and \(L_{\alpha\beta\gamma\delta}\) are the components of the plane stress moduli tensor of the plate.

The bifurcation equations in (1) need to be completed with appropriate boundary conditions at cell wall boundaries of the RVE, shown for each honeycomb considered in Figure 1.

![Figure 1](image)

Figure 1: Representative volume elements (RVEs) for: (a) rectangular and (b) hexagonal honeycomb; and (c) local plate coordinate system.

At this point we invoke the Bloch wave representation theorem for the eigenmode, according to which the eigenmode can be put in the form:

\[
\mathbf{v}(X_1, X_2, X_3) = \mathbf{p}(X_1, X_2) \exp\left(i (\omega_1 X_1 + \omega_2 X_2 + \omega_3 X_3)\right),
\]

where \(\mathbf{v} \equiv (u_1, u_2, w)\), \(\mathbf{p}\) is a periodic function with period the RVE and the wavenumber \(\omega \equiv (\omega_1, \omega_2, \omega_3)\) with \(0 \leq \omega_1 L_1 \leq 2\pi, 0 \leq \omega_2 L_2 \leq 2\pi\) and \(\omega_3 \in \mathbb{R}\).

Thanks to the Bloch wave theorem and the invariance of the principal solution along \(X_3\), all derivatives with respect to the axial local coordinate \(x_2\), equivalent to the global coordinate \(X_3\) (see Figure 1), can be substituted as \(\partial f / \partial x_2 = i\omega_3 f\), where \(f\) is a field quantity and \(\omega_3\) is the corresponding Bloch wavenumber in \(X_3\). Within each cell wall, the partial differential
equations transform into ordinary differential equations with respect to \( x_1 \), which can be solved analytically, yielding the following expressions for \( w \) and \( u \):

\[
\begin{align*}
w &= A_1 e^{q_1 \omega_3 x_1} + A_2 e^{-q_1 \omega_3 x_1} + A_3 e^{q_2 \omega_3 x_1} + A_4 e^{-q_2 \omega_3 x_1} \\
u_1 &= B_1 e^{3x_1} + B_2 e^{-\omega_{31} x_1} + B_3 x_1 e^{\omega_{31} x_1} + B_4 x_1 e^{-\omega_{31} x_1} \\
u_2 &= \left( B_1 + \frac{3 - \nu}{(1 + \nu) \omega_3} B_3 \right) e^{\omega_{31} x_1} + \left( -B_2 + \frac{3 - \nu}{(1 + \nu) \omega_3} B_4 \right) e^{-\omega_{31} x_1} + B_3 x_1 e^{\omega_{31} x_1} + B_4 x_1 e^{-\omega_{31} x_1}
\end{align*}
\]

where \( q_1 \) and \( q_2 \) can be either real or imaginary constants satisfying the equation:

\[
q^4 - \left( 2 + \frac{\sigma_{11}}{D(\omega_3)^2} \right) q^2 - \frac{i \sigma_{12}}{D(\omega_3)^2} q + \frac{\sigma_{22}}{D(\omega_3)^2} + 1 = 0 .
\]

The response of each cell wall has therefore 8 unknowns, which need to be determined applying the Bloch wave periodicity conditions, which relate the kinematics and reaction forces in opposite ends of the RVE, as well as displacement continuity and force/moment equilibrium in the central nodes.

### 2.2 Loading

The applied loading is a combination of compression in the out-of-plane direction, as well as out-of-plane shear. The lateral expansion is either allowed or fully constrained. The principal solution in all cases is a uniform state of in-plane stress \( \sigma_{\alpha \beta} \). The applied strains on the structure, expressed in the global coordinate system \( X_i \), are:

\[
\begin{align*}
\epsilon_{33} &= -\lambda , \\
\gamma_{31} &= \lambda \tan \alpha \cos \theta , \\
\gamma_{32} &= \lambda \tan \alpha \sin \theta ,
\end{align*}
\]

where \( \lambda > 0 \), \( \alpha \) describes the axial to shear strain mixity, i.e. \( \tan \alpha \) gives the ratio between applied axial and shear strains, and \( \theta \) the shear orientation, i.e. \( \tan \theta \) represents the ratio of the shear components along the \( X_1 - X_2 \) directions.

### 2.3 Onset of instability

Once a loading path characterized by \( \lambda \geq 0 \) is defined by the pair \( (\alpha, \theta) \) it is possible to assemble a matrix \( M(\lambda; \omega_1, \omega_2, \omega_3) \), which represents the eigenvalue system of equations obtained from the problem boundary conditions. For a fixed \( \omega \) the corresponding minimum buckling load parameter \( \lambda_m(\omega_1, \omega_2, \omega_3) \) is defined as the lowest \( \lambda \) root of:

\[
\text{Det} \left( M(\lambda, \omega) \right) = 0 \quad \text{(for fixed } \omega = (\omega_1, \omega_2, \omega_3) \text{)}.
\]

The critical load parameter \( \lambda_c \) is defined as the infimum of \( \lambda_m \) for all values of \( \omega \):

\[
\lambda_c = \inf_{\omega} \lambda_m(\omega) .
\]

The function \( \lambda_m(\omega) \) is continuous over all the frequency space except at the origin \( \omega = 0 \), since that point corresponds to two different types of instability modes. The first type consists of all strictly periodic eigenmodes. The second case consists of eigenmodes with wavelengths much larger than the unit cell, \( \omega \to 0 \). The presence of physically very different eigenmodes at the neighborhood of \( \omega = 0 \) explains the possibility of a singularity of \( \lambda_m \) at \( \omega = 0 \), and in case where the critical load corresponds to \( \omega \to 0 \) the use of infimum in (7).
3 Results

In this section we will use two critical strains to plot results in a physically meaningful dimensionless way. The first one is the critical strain of a compressed infinite strip with simply supported boundary conditions, \( \epsilon_{\text{strip}}^c = \frac{\pi^2 t^2}{(3(1 - \nu^2)) L^2} \). The reason for using \( \epsilon_{\text{strip}}^c \) is the fact that this is the value of the critical strain for axial compression of the rectangular honeycomb with square section and the isotropic section hexagonal honeycomb. In the case of transverse shear without axial compression we will use the critical shear strain of an infinite strip with clamped boundary conditions, \( \gamma_{\text{strip}}^c = 1.498\pi^2 t^2 ((1 - \nu) L^2) \).

3.1 Rectangular honeycomb

For a square honeycomb under only axial compression, allowing free lateral expansion, the critical mode corresponds to an antiperiodic mode with the same out-of-plane and in-plane wavelengths, with Bloch wave frequencies \( \omega_1 L = \omega_2 L = \omega_3 L = \pi \). The buckling load coincides exactly with that of an infinite simply supported strip with the same thickness \( t \) and width \( L \). The reason is that equal rotation of all plates joined along a common line is equivalent to a simply supported boundary condition for each plate. If the aspect ratio \( r = L_1/L_2 \) increases, the critical strain becomes lower, since it is determined by the wider plate, as seen in Figure 2a. The instability is still an in-plane antiperiodic mode, but the critical out-of-plane wavelength increases with \( r \). If the lateral expansion is constrained, the plates also carry in-plane stress \( \sigma_{11} \), due to Poisson’s effect. The critical mode is now a long wavelength mode, with \( \omega_3 \to 0 \). The corresponding critical buckling strains as a function of \( r \) are shown in Figure 2a. As expected from the general theory [1], the results coincide with those obtained analyzing the loss of ellipticity in the homogenized structure.

The effect of transverse shear loading has been studied for two different geometries, \( r = 1 \) and \( r = 2 \). Free lateral expansion is allowed in both cases. The results for only shear loading are shown in Figure 2b. For the case of a square section honeycomb (\( r = 1 \)) the critical strain graph is symmetric about \( \theta = \pi/4 \), in view of symmetry of the cross-section. As the aspect ratio increases, the symmetry of the graph about \( \theta = \pi/4 \) is destroyed. In the case of combined axial compression and transverse shear, the presence of shear further destabilizes the cell walls and hence decreases the critical axial strain, as seen in Figure 2c for the case of \( r = 2 \) and four different axial to shear mixity angles \( \alpha \).
3.2 Hexagonal honeycomb

The critical mode for the axially compressed is in-plane periodic, $\omega_1 L = \omega_2 L = 2\pi$, with $\omega_3 L = \pi$ and $\epsilon_{33} = \epsilon_{\text{strip}}^c$. The reason is the same as in the square section honeycomb: equal rotation of the plates joined along a common line results in zero bending moment, which is equivalent to simply supported boundary conditions. The anisotropic-section honeycomb has a higher critical strain, $\epsilon_{33}^c = 1.55\epsilon_{\text{strip}}^c$, due to the presence of the double thickness cell wall. The out of plane eigenmode frequency is $\omega_3 L = 1.26\pi$.

The case of axial compression with constrained lateral expansion can only been studied for the isotropic-section honeycomb. The current method cannot be used for the laterally constrained anisotropic-section honeycomb since in-plane equilibrium imposes bending of the cell walls in the principal solution. The effect of constraining the lateral expansion for the axially compressed, isotropic-section honeycomb is similar to that on its rectangular counterpart: due to the additional compressive stresses $\sigma_{11}$, the critical strains are reduced and a long wavelength mode appears in $X_3$. The critical strain is $\epsilon_{33}^c = 0.2497\epsilon_{\text{strip}}^c$, and there are three possible critical eigenmode wavenumbers, $(\omega_1 L_1, \omega_2 L_2) = (\pi, \pi), (\omega_1 L_1, \omega_2 L_2) = (\pi, 2\pi)$ and $(\omega_1 L_1, \omega_2 L_2) = (2\pi, \pi)$, which are related by a rotation of $\pi/3$ radians, due to the symmetry in the structure.

The critical strain under transverse shear as function of the shear orientation is shown for the isotropic-section hexagonal honeycomb and the anisotropic-section hexagonal honeycomb in Figure 3a. Notice that for the isotropic-section honeycomb the critical strain is rather insensitive to the load angle $\theta$, with a maximum difference of the order of about 2.5%, while for the anisotropic-section honeycomb the difference is about 40%. As expected from the existence of double thickness walls, the critical strain for the anisotropic-section honeycomb is always higher than for its isotropic-section counterpart.

For combined axial compression and transverse shear, there is again a reduction in the critical axial load, as found for the rectangular honeycomb case. Results of four different axial to shear mixity angles $\alpha$ have been calculated both for the isotropic-section case in Figure 3b and the anisotropic-section case in Figure 3c.

Figure 3: Critical strain for a hexagonal honeycomb under: (a) out-of-plane shear with free lateral expansion, (b) combined out-of-plane compression and shear with free lateral expansion for isotropic-section, and (c) combined out-of-plane compression and shear with free lateral expansion for anisotropic-section.

4 Conclusions

The present work pertains to the onset of a bifurcation (buckling type) instability in axially compressed and transversally sheared perfectly periodic honeycomb of infinite extent. The critical load and corresponding eigenmodes are found analytically using a Bloch wave representation of
the eigenmode by considering only the smallest unit cell for different rectangular and hexagonal geometries, since they are the most frequently used in applications.

Due to the symmetry of loading and geometry, all cell walls remain flat in the principal solution of the 3D problem at hand. Given the experimental results in axially compressed thin wall honeycomb by [3], all calculations reported here are done in the elastic regime of the cell wall’s response, although a generalization to the case of thick wall honeycomb that requires plastic constitutive description is straightforward and easily fits the framework of the present model.

Simulations for the bifurcation equilibrium in axially crushed honeycomb [3] show a stable, supercritical post-bifurcation equilibrium path that requires additional straining before reaching a maximum load that appears after the cell wall material enters the plastic range of its material response. Consequently, the onset-of-bifurcation results presented here are expected to be lower bounds for the critical loads of an actual, imperfect, finite-size structure.

The main contribution of this work is, in addition to the evaluation of a lower bound for the critical load, a consistent calculation of the corresponding eigenmode for the infinite, perfect honeycomb, thus providing an estimate of a minimum representative volume needed for numerical calculations in crushing simulations. It is worth pointing out that the critical eigenmode depends strongly on the applied load orientation. It should also be mentioned at this point that in technological applications the plateau for the crushing is of paramount importance, but unlike the the onset of bifurcation its calculation requires the evaluation of a complicated post-bifurcated equilibrium path where plasticity, contact, friction and imperfections play a determinant role, thus adding to the appeal of the – relatively much easier – onset-of-bifurcation calculations presented here.

Acknowledgements

Support for this study by the PSA Chair André Citroën is gratefully acknowledged.

References

