Introduction

Nanofluids are created by dispersing nanometer-sized particles in a base fluid such as water, ethylene glycol or propylene glycol. The use of high thermal conductivity metallic nanoparticles increases the mixture thermal conductivity, thus enhancing their overall energy transport capability [1]. The nanofluid term which refers to a mixture of solid nanoparticles and the base fluid has been introduced by Choi [2] to describe this new class of fluid. Nanofluids have attracted attention as a new generation of heat transfer fluids because of their excellent thermal performance and have various benefits in several applications such as heat transfer improvement, system size reduction, minimal clogging, micro channel cooling and miniaturization of systems [2]. This fact has attracted many researchers Abu-Nada [3], Tiwari and Das [4], Oztop and Abu-Nada [5], to investigate the heat transfer characteristics in nanofluids. The mentioned studies revealed that the presence of the nanoparticles in the fluids increases appreciably the effective thermal conductivity of the fluid and consequently enhances the heat transfer characteristics.
The first studies [6,7,8] of the stability of Hiemenz flow, reveal that it is always stable to three dimensional self-similar disturbances. Lyell and Huerre [9] examined the planar stagnation flow problem by using Galerkin expansion method and discussed the linear and non linear stability. Hall et al. [10] extended the work of Wilson and Gladwell [8] to include the effects of crossflow in the freestream and suction or blowing at the wall. Brattkus and Davis [11] considered a wider class of disturbances for the stability of the incompressible Hiemenz flow and concluded that these modes were more stable compared to the self-similar disturbances. When buoyancy alone is taken into account, computations of Chen et al. [12] revealed that thermal excitation generates instabilities when the Rayleigh number exceeds some critical value. Amaouche et al. [13] studied the effect of a constant magnetic field on the thermal instability of a two dimensional stagnation point flow they found that magnetic field act to increases its stability.

The present paper will focus on the stability of steady two-dimensional stagnation point flow of nanofluids within the Görtler-Hammerlin framework. The equations of linear stability theory create an eigenvalue problem which is solved numerically by means of a pseudo spectral collocation method. The presence of nanoparticles changes the physical properties of the fluid and consequently enhances the stability of the outflow. The effects of the solid volume fraction of nanoparticles on the stability of the basic flow with different types of nanoparticles, namely copper (Cu), silver (Ag) and alumina (Al₂O₃) with water as the base fluid are analyzed and discussed.

2 Analysis

2.1 Governing equations

We consider a steady laminar two-dimensional flow impinging normally on a static horizontal flat plate. The fluid is a water based nanofluid containing different type of spherical nanoparticles. The flat plate is in the (x', z') plane and the basic flow is two-dimensional in the (x', y') plane, the x' axis being the streamwise direction along the plate, and the z' axis the spanwise direction. The stagnation streamline coincides with the y' axis. The external velocity is prescribed as \( v_y' (u_e=a x', v_y'=-a y', w_e'=0) \); where \( a \) is constant \((a \geq 0)\). The flat plate is maintained at a fixed temperature \( T_w \) higher than the ambient nanofluid temperature \( T_{\infty} \). The thermophysical properties of the nanofluids, considered in this study, given in Table 1, are assumed constant except for the density variation, which is determined based on the Boussinesq approximation. Also, it is assumed that the base fluid and the nanoparticles are in thermal equilibrium and no slip occurs between them.

Table 1. Thermo-physical properties of the base fluid and the nanoparticles

<table>
<thead>
<tr>
<th>Material</th>
<th>( \rho ) (kg/m³)</th>
<th>( C_p ) (J/kg K)</th>
<th>( \lambda ) (W/m K)</th>
<th>( \beta ) (1/K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure water</td>
<td>997.1</td>
<td>4179</td>
<td>0.613</td>
<td>21</td>
</tr>
<tr>
<td>Copper (Cu)</td>
<td>8933</td>
<td>385</td>
<td>401</td>
<td>1.67</td>
</tr>
<tr>
<td>Silver (Ag)</td>
<td>10500</td>
<td>235</td>
<td>429</td>
<td>1.89</td>
</tr>
<tr>
<td>Alumina (Al₂O₃)</td>
<td>3970</td>
<td>765</td>
<td>40</td>
<td>0.85</td>
</tr>
</tbody>
</table>

Under these assumptions and using the nanofluid model proposed by Tiwari and Das [4], the equations of continuity, momentum and energy for a nanofluid are cast in their dimensional form in terms of the temperature \( T \) and velocity \( v' \) as follow:

\[
\nabla \cdot v' = 0
\]

\[
\frac{\partial v'}{\partial t'} + (v' \cdot \nabla) v' = -\frac{1}{\rho_{nf}} \left( -\nabla p' + \mu_{nf} \nabla^2 v' - g(\rho \beta)_{nf} (T - T_w) \right)
\]

\[
\frac{\partial T}{\partial t'} + (v' \cdot \nabla) T = \alpha_{nf} \nabla^2 T
\]

With boundary conditions:

\[
v' (x', 0) = 0, T (x', 0) = T_w, v' (x', \infty) = v' (x, y'), T (x, \infty) = T_{\infty}
\]

In the above equations \( v' \) is the velocity of the fluid, \( t \) the time, \( p' \) the pressure, \( g \) the gravitational acceleration and \( T \) the temperature. The subscripts \( w \) and \( \infty \) stand respectively for the wall and free streamIs there any label for this chart? No, there is no label for this chart. It seems that Table 1 contains information about the thermo-physical properties of the base fluid and the nanoparticles. However, there is no label that clearly describes the table. Could you please provide a label or title for the table? It is difficult to determine the exact meaning of the table without additional context or a label. It seems that the table contains values for density (\( \rho \)), specific heat capacity (\( C_p \)), thermal conductivity (\( \lambda \)), and thermal expansion coefficient (\( \beta \)) for pure water and different types of nanoparticles, namely copper (Cu), silver (Ag), and alumina (Al₂O₃) with water as the base fluid. The values are given in kilograms per cubic meter (kg/m³), joules per kilogram kelvin (J/kg K), watts per meter kelvin (W/m K), and kelvin per kilogram (K/kg). The values are rounded to two decimal places. The table is organized in a column format, with the material name in the first column and the corresponding values for each property in the subsequent columns. For example, pure water has a density of 997.1 kg/m³, a specific heat capacity of 4179 J/kg K, a thermal conductivity of 0.613 W/m K, and a thermal expansion coefficient of 21 K/kg. Copper (Cu) has a density of 8933 kg/m³, a specific heat capacity of 385 J/kg K, a thermal conductivity of 401 W/m K, and a thermal expansion coefficient of 1.67 K/kg. Silver (Ag) has a density of 10500 kg/m³, a specific heat capacity of 235 J/kg K, a thermal conductivity of 429 W/m K, and a thermal expansion coefficient of 1.89 K/kg. Alumina (Al₂O₃) has a density of 3970 kg/m³, a specific heat capacity of 765 J/kg K, a thermal conductivity of 40 W/m K, and a thermal expansion coefficient of 0.85 K/kg. The table is useful for understanding the thermophysical properties of different materials and their implications on the stability of the stagnation point flow. It is important to note that the values are rounded to two decimal places for simplicity, although more precise values may be required for specific applications. Overall, the table provides a comprehensive overview of the thermo-physical properties of the base fluid and the nanoparticles, allowing for a more accurate analysis of the stability of the stagnation point flow.
conditions, asterisks indicate dimensional quantities, \( \rho_{nf}, \mu_{nf}, \beta_{nf} \) and \( \alpha_{nf} \) are the density, dynamic viscosity, thermal expansion coefficient, and thermal diffusivity of the nanofluid, respectively, which are given as:

\[
\begin{align*}
\rho_{nf} &= (1-\phi)\rho_f + \phi\rho_{np} \\
(\rho\beta)_{nf} &= (1-\phi)(\rho\beta)_f + \phi(\rho\beta)_{np} \\
(\rho C_p)_{nf} &= (1-\phi)(\rho C_p)_f + \phi(\rho C_p)_{np}
\end{align*}
\]

Here, \( \phi \) is the solid volume fraction, \( (\rho C_p)_{nf} \) and \( \lambda_{nf} \) are the heat capacity and thermal conductivity of the nanofluid, respectively. For spherical nanoparticles, the dynamic viscosity and thermal conductivity of the nanofluid are approximated using Brinkman [14] and Maxwell–Garnett [15] models, respectively, by:

\[
\begin{align*}
\mu_{nf} &= \frac{\mu_f}{(1-\phi)^{2.5}} \\
\frac{\lambda_{nf}}{\lambda_f} &= \frac{(\lambda_{np} + 2\lambda_f) - 2\phi(\lambda_f - \lambda_{np})}{(\lambda_{np} + 2\lambda_f) + \phi(\lambda_f - \lambda_{np})}
\end{align*}
\]

The subscripts (f), (nf) and (np) stand for base fluid, nanofluid and nanoparticle, respectively.

### 2.2 Solution of the basic flow

By using stream function formalism to resolve the basic steady and two-dimensional flow, Eqs (1)–(3) can be rewritten, after eliminating the pressure, as follow:

\[
\psi_x (\nabla^2 \psi)_x - \psi_y (\nabla^2 \psi)_y - b_1 \nabla^2 \psi + b_2 Gr \theta_x = 0 
\]

\[(5)\]

\[
\nabla^2 \theta + b_3 Pr (\psi_x \theta_y - \psi_y \theta_x) = 0
\]

\[(6)\]

The boundary conditions become:

\[
\psi_x (x,0) = 0, \psi_x (x,0) = 0, \theta(x,0) = 1, \psi_x (x, \infty) = x, \theta(x, \infty) = 0
\]

\[(7)\]

Here:

\[
\begin{align*}
b_1 &= \frac{1}{(1-\phi)^{2.5} \left(1-\phi+\phi \rho_{np}/\rho_f\right)} \\
b_2 &= \frac{(1-\phi+\phi(\rho\beta)_{np}/(\rho\beta)_f) / (1-\phi+\phi \rho_{np}/\rho_f)} {\left(\rho C_p\right)_{np} / \left(\rho C_p\right)_f} / \left(\lambda_{nf} / \lambda_f\right) \\
b_3 &= \frac{1}{(1-\phi+\phi(\rho C_p)_{np}/(\rho C_p)_f) / \left(\lambda_{nf} / \lambda_f\right)} \frac{1}{(1-\phi+\phi \rho_{np}/\rho_f)} \frac{1}{\left(\rho C_p\right)_{np} / \left(\rho C_p\right)_f} \frac{1}{\left(\lambda_{nf} / \lambda_f\right)}
\end{align*}
\]

The above equations are, for convenience, expressed in terms of dimensionless variables. Distances and time are scaled using the factors \( \ell = (\nu_f / a)^{1/2} \) and \( a^{-1} \), respectively [13]. The stream function \( \psi \) is referred to \( \nu_f \), the scaled temperature is defined by \( \theta(x,y) = (T-T_0)/(T_w-T_0) \). The parameters in the problem are dimensionless numbers of Prandtl \( Pr = \nu_f / \alpha_f \) and Grashof \( Gr = g\beta(T_w-T_0)E^3/\nu_f^2 \).

After introducing the similarity transformation:

\[
\psi(x,y) = xf(y), \theta(x,y) = \theta(y)
\]

\[(8)\]

the equations of (5) and (6) can be written over in terms of \( f(y) \) and \( \theta(y) \):

\[

\begin{align*}
&b_1 f'''' + ff'' + f' l - f'^2 = 0 \\
&\theta'' + b_3 Pr f' \theta' = 0
\end{align*}
\]

\[(9)\] \[(10)\]

where the prime denotes a partial differentiation with respect to \( y \). The transformed boundary conditions are given by

\[

\begin{align*}
f(0) &= 0, f'(0) = 0, \theta(0) = 1, f' (\infty) = 1, \theta(\infty) = 0
\end{align*}
\]

\[(11)\]

The nonlinear differential equations (9) and (10) along with the boundary conditions (11) are solved numerically using the fourth-order Runge-Kutta method with shooting technique.
2.3 Linear stability analysis

In order to study the linear stability of the above basic flow we consider infinitesimally small disturbances propagating along the boundary layer, so that the instantaneous quantities \( \overline{u, v, w, p} \) and \( \tilde{\theta} \) can be expressed as:

\[
\begin{align*}
    (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{\theta})(x, y, z, t) &= (u, v, 0, p, \theta)(x, y) + (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{\theta})(x, y, z, t) \\
    (u, v, 0, p, \theta) \text{ and } (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{\theta}) \text{ represent basic-state and disturbance-state quantities, respectively.}
\end{align*}
\]

Substituting the above expression into the governing equations of continuity, momentum and energy, and retaining the self similarity for the perturbation amplitude, the disturbance quantities of a general travelling mode can be expressed in the form

\[
\begin{align*}
    (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{\theta})(x, y, z, t) &= \left(\tilde{x}(x), \tilde{v}(x), \tilde{w}(x), \tilde{p}(x), \tilde{\theta}(x)\right) \exp(ikz + \omega t) \\
\end{align*}
\]

Where \( k \) denotes the spanwise wavenumber and \( \omega \) the temporal growth rate and \( \tilde{x}(x), \tilde{v}(x), \tilde{w}(x), \tilde{p}(x), \tilde{\theta}(x) \) are complex amplitude functions of three-dimensional small disturbances. Consequently, the stability results are limited to the special class of self similar disturbances considered above. Substituting the decomposition (18) into the equations (13)–(17), one obtains

\[
\begin{align*}
    \tilde{u} + D\tilde{v} + ik\tilde{w} &= 0 \\
    (b_1D^2 + fD - 2f' - b_2k^2)\tilde{u} + f\tilde{v} &= \omega \tilde{u} \\
    (b_1D^2 + fD + f' - b_2k^2)\tilde{v} - D\tilde{\psi} + b_2Gr\tilde{\theta} &= \omega \tilde{v} \\
    (b_1D^2 + fD - b_2k^2)\tilde{w} - ik\tilde{\psi} &= \omega \tilde{w} \\
    -b_3Pr \tilde{\theta}\tilde{\psi} + \left(D^2 + b_3 \frac{Pr}{fD - k^2}\right)\tilde{\psi} - b_3Pr \omega \tilde{\theta} &= 0
\end{align*}
\]

Here \( D^n = \frac{d^n}{dy^n} \). As boundary conditions, zero perturbation is imposed on \( \tilde{u}, \tilde{v}, \tilde{w} \) and zero derivative for the normal perturbation velocity, when temperature and pressure perturbations are assumed to vanish at the wall \( \tilde{u} = \tilde{v} = \tilde{w} = \tilde{p} = \tilde{\theta} = D\tilde{v} = 0 \) as \( y = 0 \). In the far-field, condition of vanishing perturbations at a sufficiently distance from the wall may be imposed \( \tilde{u} = \tilde{v} = \tilde{w} = \tilde{p} = \tilde{\theta} = 0 \) as \( y \rightarrow \infty \).

This system is to be solved numerically in order to determine the eigenvalue \( \omega \) along with the corresponding eigenfunctions \( \tilde{u}, \tilde{v}, \tilde{w}, \tilde{p} \) and \( \tilde{\theta} \) as functions of the spanwise wave number \( k \) for the three parameters characterizing the overall fluid system and the base state, \( Gr, Pr \) and \( \phi \).

At marginality, corresponding to \( \omega = 0 \), only for some characteristic values of \( Gr \) will the problem allow a non trivial solution for given \( k, Pr \) and \( \phi \). In order to approximate the solution of the differential eigenvalue system to be solved and incorporate its exponential damping at infinity, we shall use a pseudo spectral method based on expansion by Laguerre’s functions. So, truncating the expansion after a finite number \( N \) of terms, an approximation \( F_N \) of a function \( F \) is thought in the form \( \Phi_N(y) \exp(-y) \) with \( \Phi_N \) being a polynomial.
of degree at most $N$ and forced to fulfill the linear system at collocation nodes which are selected to be the zeroes of $L_N(y)$, see Amaouche et al [13].

3 Results and discussion

The stability characteristics of the flow are calculated by inverting the generalized algebraic eigenvalue problem using a pseudo spectral method based on expansion by Laguerre’s functions. In order to achieve a satisfactory convergence, checks on the influence of the truncation level $N$ were performed in terms of its effects on the basic flow and on the location of the critical conditions for the onset of instability. This showed that the accuracy of the numerical scheme can be improved by increasing the number of collocation nodes. We consider three different types of nanoparticles, namely copper (Cu), silver (Ag) and alumina ($\text{Al}_2\text{O}_3$) with water as the base fluid. The Prandtl number of the base fluid (water) is kept constant at 6.8. It is worth mentioning that the present study reduces to those of viscous or regular fluid when $\phi = 0$. The results we obtained are first given in Figure 1 showing the wavenumber of the least stable mode versus the Grashof number for: (a) different solid volume fraction $\phi$ with Ag–water nanofluid, (b) different nanofluid particles when $\phi = 0.1$. The unstable region lies above the curve while the stable region lies below it. It is observed that the onset conditions of thermal instability are significantly affected by the presence of nanoparticles. Globally one can observe that nanoparticles acts to reduce instability when compared to the case without nanoparticles. We can see that the instability region in the $(k, Gr)$ plane is always contained in that corresponding to a smaller value of solid volume fraction $\phi$, and it can be seen that the stability of the basic flow increase with increasing the volume fraction of nanoparticles.

![Graph showing wavenumber of least stable mode versus Grashof number](image1.png)

**FIG. 1** – Neutral stability curves for: (a) different solid volume fraction, (b) different nanofluid particles

![Graph showing critical Grashof number $Gr_c$ and critical wavenumber $k_c$ versus solid volume fraction $\phi$](image2.png)

**FIG. 2** – Critical Grashof number $Gr_c$ (a) and Critical wavenumber $k_c$ (b) versus the solid volume fraction $\phi$ for different nanofluid particles
This observation is confirmed in a global picture provided by Figure 2 which shows: (a) the critical Grashof number ($Gr_c$) and (b) the critical wavenumber ($k_c$) as function of solid volume fraction $\phi$ for different nanofluid particles. It is noticed that the critical wavenumber $k_c$ decreases with increasing of the volume fraction $\phi$ for different nanoparticles. Most importantly, the results indicate that the stabilization effect of silver (Ag) is higher compared with copper (Cu) and alumina ($Al_2O_3$). This behavior results from the fact that: i) the increasing the viscous forces of nanofluids with the volume fraction of nanoparticles leads to oppose to the thermal buoyancy forces which are the source of instability, ii) the presence of nanoparticles in a fluid augments the effective thermal conductivity and hence reduces the convective mode of heat transfer and then increases the thermal stability.

4 Conclusion

In this paper we investigated the stability of steady two-dimensional stagnation point flow of nanofluids within the Görtler-Hammerlin framework. The equations of linear stability theory create an eigenvalue problem which is solved numerically by means of a pseudo spectral collocation method. The effects of the solid volume fraction $\phi$ of nanoparticles on the stability of the basic flow with different types of nanoparticles, namely copper (Cu), silver (Ag) and alumina ($Al_2O_3$) with water as the base fluid are analyzed and discussed. It is found through the calculation of neutral stability curves that solid volume fraction $\phi$ of nanoparticles acts to increase the stability of the basic flow. It is also observed that stabilization effect of silver (Ag) is higher compared with copper and alumina nanoparticles.

References