Compaction of thin sheets: crumpling and folding

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Résumé :
L’ADN dans les capsides virales ou les feuilles dans les bourgeois sont des exemples biologiques d’objets minces confinés dans un contenant. Tiges et plaques préfèrent se courber ou à défaut localiser les déformations comme dans une boulette de papier froissé. Nous réalisons des expériences de papier froissé pour étudier l’apparition des déformations à petite échelle lorsque le confinement est imposé à grande échelle. Au cours du froissement d'une feuille de papier, nous mesurons la force et caractérisons la géométrie en configuration pliée par ‘tomographie’ manuelle, que nous comparons à celles attendues pour du papier plié régulièrement : de remarquables analogies apparaissent entre papier froissé et papier plié [1].

Abstract :
DNA packaging in viral capsids or plant leaves growing in buds are some biological examples of close-packed low dimensional structures. Rods and plates preferentially bend or alternatively localize strains like in a crumpled paper ball. We experimentally study crumpled sheets to investigate how small scale patterns emerge when confinement is imposed at large scale. During the crumpling of a paper sheet, force is measured and geometry in crumpled configuration is characterized thanks to hand-made ‘tomography’, that we compare to predictions developed for regularly folded paper: we show that simple folding models allow us to capture also the main features of crumpling [1].

Mots clefs : folding, crumpling, compaction, elasticity, bending, experiments, statistical mechanics

FIG. 1 – Leaves of flowers growing inside confining buds are some examples of close-packed low dimensional biological structures: poppy, lotem, datura.

1 Introduction
A seemingly simple packing problem is how to pack efficiently a quasi two-dimensional sheet into a three-dimensional volume. This problem is relevant for biology: flower buds for instance can exhibit crumpled shapes, although they can also have a cabbage-like lamellar structure [2,3] (Figure 1). Crumpling also has many applications in physics and engineering - with handling of paper waste being one of these. Crumpled morphologies often result from application of stresses or strains via the boundaries, but compaction can also be achieved thorough a confining potential or residual stresses, such that the forces can act on the whole
volume rather than just on the boundaries of it [4-8]. Crumpled configurations of materials are hard to describe theoretically, due to highly non-linear strains, possible plastic deformation, friction, jamming in metastable states and more. In view of this there is currently no solid theoretical work on crumpled thin sheets under strong confinement. In the case of packing, previous experimental [9-11] and numerical [4,5,12,13] studies focused on the geometrical properties of ridges and facets after uncrumpling or under weak confinement, leading to various descriptions of their lengths statistics. More recently, three-dimensional x-ray tomography was used to study isotropic crumpling and found a power-law between the crumpling force and the compaction strength when crumpling at constant force [14,15]. A similar relationship was found when crumpling at controlled compaction [11], and power-laws are common in packing problems [4,11-12,14-15,35-39]. Here we aim at a parallel experimental study of geometry and forces of crumpled configurations, allowing to understand their relationship. We investigate properties of ridges and facets in crumpled samples for a range of compaction strengths, in an attempt to highlight some properties of the crumpling mechanism. We show that simple folding models allow us to capture also the main features of crumpling. The present proceeding is aimed at detailing the geometrical aspects, whereas the whole work is published in [1].

2 Experiments of crumpling: Geometry

For the geometry, we use an original approach to study crumpling using properties of ridges and facets in crumpled samples, and more specifically by looking at cross-sections. Sheets of different paper types and sizes are crumpled into a (handmade) ball or into a (machine-made) pancake at different compaction strengths. The compaction strength is quantified by the ratio \( d/D \), with \( d \) and \( D \), the initial and final sizes of the sphere surrounding the uncrumpled and crumpled sheets respectively. A cross-section is obtained by cutting the crumpled ball/pancake with a slowly moving hot wire (a technique used originally in [16]). From cross-sections (Figure 2), the number of layers is measured in two orthogonal directions passing...
through the center, then averaged to get the mean number of folds in the crumpled ball. The cut crumpled sheet is reopened carefully. Several uncrumpled pieces, with possibly several holes, are obtained and scanned (Figure 3). The edges of their boundaries and holes are detected automatically, broken down into branches delimited by kinks, or more generally by points of high curvature, and referred to as segments as in [4]. For the segmentation, we use a ‘split and merge’ algorithm, which uses a threshold distance $d_{\text{thresh}}$ that controls its precision. Dedicated experiments were performed to determine the relevant value for $d_{\text{thresh}}$ resulting in $10h$, where $h$ is the thickness of the sheet of paper. Later we will show that our results are not sensitive to the choice of this parameter. Such planar two-dimensional cross-sections of a crumpled sheet bear information on the full three-dimensional crumpled configuration. In particular, the ensemble of segments samples the facets delimited by ridges, so that its length distribution is related to some characteristic distances between ridges or equivalently to some characteristic sizes of the facets. Let $p(x)$ be the probability density function of the characteristic linear dimension $x$ of facets. In these terms, our cross-sectional sampling allows us to measure the pdf $\Pi(\ell) = \int p(x) f(\alpha) \delta(\ell - \alpha) d\alpha d\alpha$, where $\delta$ is the Dirac function and $f(\alpha = \ell / x)$ describes the sampling process. From the previous formula, we see that the measured pdf $\Pi(\ell)$ gives some information on $p(x)$ if $f(\alpha)$ is peaked around a particular value - which is the case for circular, square, quadrilateral with isotropic facets, and more generally for regular facets with strictly more than 3 edges. Therefore, lengths of segments are measured and statistically analyzed.

$$\int_{-\infty}^{\infty} = \frac{1}{\beta} \left( \frac{x}{\beta} \right)^{\alpha - 1} e^{-\left( \frac{x}{\beta} \right)^{\alpha}} dx$$

Figure 4a shows the probability density function of segments' length $\rho(\ell)$, log-normal law $f_{LN}$ (continuous line) and Gamma distribution $f_{\Gamma}$ (dashed line) with the same mean and variance on log-log scales for several sets. (b) Table showing their respective errors $\chi^2(f) = \sum_{i=1}^{N} \left( \rho(i) - f(i) \right)^2 / f(i)$, with $\rho$ the experimental pdf and $f$ the reference pdf. (c) The mean segment size as a function of its standard deviation.

FIG. 4 – (a) Probability density function of segments' length $\rho(\ell)$, log-normal distribution $f_{LN}$ (continuous line) and Gamma distribution $f_{\Gamma}$ (dashed line) with the same mean and variance on log-log scales for several sets. (b) Table showing their respective errors $\chi^2(f) = \sum_{i=1}^{N} \left( \rho(i) - f(i) \right)^2 / f(i)$, with $\rho$ the experimental pdf and $f$ the reference pdf. (c) The mean segment size as a function of its standard deviation.

FIG 5. – (a) The mean length of the segments as a function of the compaction strength $d/D$ for different materials and sizes. (b) Correlation coefficient of the length of 2 segments separated by $n$ segments.

Figure 4a shows the probability density function of segments' length, superimposed on a log-normal law and
a Gamma law (Equation 1), with the mean and variance matching the experimental values. The two characteristic length-scales, namely the mean and the standard deviation are not independent, and are actually proportional to each other, and are very close (Figure 4c). This implies that there is actually only one independent characteristic length-scale in this problem.

\[
f_{\text{LN}}(\ell) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left( -\frac{(\ln(\ell) - \mu)^2}{2\sigma^2} \right)
\]

\[
f_{\Gamma}(\ell) = \frac{\ell^{\alpha-1}}{\Gamma(\alpha)\theta^\alpha} \exp\left( -\frac{\ell}{\theta} \right)
\]

Both distributions (Equation 1) describe rather well the experimental data. However, the log-normal law is the only one that describes well the probability density function for short segments: the decreasing probability for decreasing lengths appears to be faster than a power law, as would have been expected for a Gamma law. A more rigorous test is possible through the statistical \(\chi^2\) test for goodness-of-fit, which confirms that the log-normal describes better the data (Figure 4b). We checked that this description is robust with respect to the chosen value of the threshold \(d_{\text{thresh}}\) used in the segmentation procedure, including wide variations of \(d_{\text{thresh}}\). As a whole, the experimental distributions are closer to a log-normal law, and this holds for all the experimental data sets. From a theoretical point of view, a log-normal distribution is found in a hierarchical fragmentation process - when an initial segment is cut into two pieces at random, and then repeatedly, at each step, all segments are cut into two pieces according to the same random rule. The central limit theorem ensures that the probability distribution function of segment sizes converges to a log-normal distribution [4,26-27] in that case.

On the other hand a Gamma distribution is expected when the buckling process is no longer hierarchical [4,28-29]. A simple way to see this is the following - A Gamma law with \(\alpha = 1\) (see Eq. 1) is actually equivalent to an exponential distribution of lengths, which describes the fragment distribution of a line segment [0,1] where cracks are introduced at random on it. To be more precise, if the distribution of cracks is uniform on the segment [0,1], then the distribution of fragments is exponential [25]. Also, the Gamma law reduces to the exponential distribution when \(\alpha = 1\) [25]. A Gamma distribution with general (integer) \(\alpha\) would then describe a fragment distribution resulting from sums of \(\alpha\) such elementary fragments (or schematically, \(\Gamma = \sum \exp\)). It was suggested, in the context of fluid mixing and fragmentation [28-29], that \(\alpha\) is directly proportional to the number of interacting layers involved in creating the segment distribution seen experimentally. This was later carefully checked, and justified in Ref. [4] (see Figure 4 there), in the context of a 2D model of crumpled paper. As mentioned above, in our experiments (as in [9]) we find log-normal distributions. Also, the average length of segments decreases with increased compaction (Figure 5a): it seems to first decrease rather quickly and then to saturate for high compaction strength, but more data would be required to be more precise. All this indicates that it is likely that the balls and pancakes are obtained by a hierarchical folding of the sheet. This hierarchical crumpling leads to the replication of similar patterns at all scales during the crumpling: symmetries of the folding process are clearly visible, on cut and uncrumpled pieces (Figure 3). This could be a possible source of long-range correlations in the geometry [9-10,30-31]. Correlations are attested by the non-zero value of correlation coefficient of length of 2 segments separated by \(n\) segments from the uncrumpled cross sections (Figure 5b).

### 3 Conclusion

An original geometrical approach - based on the geometrical characterization of the crumpled paper by using the properties of folds and facets in cross sections of crumpled samples - shows that crumpling is hierarchical, just as the repeated folding. The conclusion of our whole work [1] is that folding and crumpling are very similar in nature and the crumpling process shall be viewed as arising from successive folding events. For ordered folding, simple models allow for predictions of the relations between force, compaction ratio, and number of folds. Surprisingly, these are found to capture the main properties of crumpling also, in particular
the hierarchical structure of the folds and the power law relation between the force and the compaction ratio.

References