Stochastic averaging and jet formation in planetary turbulence.

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Abstract:
Geophysical turbulent flows are characterized by their self-organisation into large scale coherent structures, in particular parallel jets. We will present a theory in order to describe the effective statistics and dynamics of these jets \cite{3}. We prove that this closure is exact in the limit of a time scale separation between the forcing and the inertial dynamics, which is rare in a turbulent flow. The equation obtained describes the attractors for the dynamics (alternating zonal jets), and the relaxation towards those attractors. At first order, these attractors are the same as the ones obtained from a quasi-Gaussian closure, already studied. Our work thus justifies this approximation and the corresponding asymptotic limit. It also goes beyond, indeed it describes the stationnary distribution of the jets (fluctuations and large deviations), and predicts the corrections to the quasi-Gaussian approximation.

Mots clefs : geostrophic turbulence; coherent structures; stochastic averaging

1 Introduction
The emergence of large-scale, long-lived, coherent structures is the main aspect of geophysical and astrophysical flows \cite{4}. The common pictures of Jupiter perfectly illustrate this fact: the surface flow is clearly organised into parallel, alternating zonal jets (parallel to the equator), with also a presence of giant and very stable vortices such as the Great Red Spot. Such large scale features are on one hand slowly dissipated, mainly due to a large-scale friction mechanism, and on the other hand maintained by the small-scale turbulence, through the Reynolds’ stress. The main mechanism is thus a transfer of energy from the forcing scale (due to barotropic and baroclinic instabilities) to the turbulent scales and until the scale of the jets. An important point in this phenomenology is the fact that the turbulent fluctuations are of very weak amplitude compared to the amplitude of the zonal jet, and that they evolve much faster. This means that the typical time scale of advection and shear of the fluctuations by the jet is much smaller that the typical time scale of formation or dissipation of the whole jet. This time scale separation is a very specific property of the geophysical large-scale structures, and it is a crucial element that will be stressed throughout the presentation.
Numerical simulations of atmosphere flows can illustrate this phenomenon (fig. 1, [7]), and it is observed that the fully non-linear dynamics is not necessary to describe the formation of jets. The so-called quasi-linear approximation seems to be sufficient in order to reproduce quantitatively the dynamics of the jets.

![Figure 1](image1.png)

**Figure 1** – Examples of direct numerical simulations of an atmospheric flow (primitive equations), from [1]. The colors represent the intensity of the Reynolds’ stress divergence (in units $10^{-6}$ m.s$^{-2}$), and the solid lines are the isolines of the zonally averaged flow (in units m.s$^{-1}$). We clearly see the formation of intense jets ($25$ m/s) in high altitudes and midlatitudes ($\sim 30^\circ$). The result of the full simulations (left panel) is in very good agreement with the simulation of the quasi-linear dynamics (right panel), particularly when it comes to the averaged velocity (solid lines).

In this turbulent context, the understanding of jet formation requires averaging out the effect of rapid turbulent degrees of freedom in order to describe the slow evolution of the jet structure. Such a task, an example of turbulent closure, is usually extremely hard to perform for turbulent flows. We prove that it can be performed explicitly in this problem. Moreover, it gives at leading order a quasi-Gaussian closure, which is naturally related to the quasi-linear dynamics presented above. The success of this approach strongly relies on the time scale separation mentioned earlier.

The simplest model that leads to the formation of such jets is the one layer barotropic equations, on a beta-plane or over a topography, with stochastic forcing:

$$\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = -\alpha \omega + \nu \Delta \omega + \sqrt{2\alpha} \eta,$$

with the non-divergent velocity $\mathbf{v} = e_z \times \nabla \psi$, the vorticity $\omega = \Delta \psi$ and the potential vorticity $q = \omega + h(y)$, where $\psi$ is the streamfunction and $h$ the topography (or $h(y) = \beta y$ in the case of the beta-plane equations). We have introduced a linear friction term $-\alpha \omega$ which describes the large-scale dissipation of energy, and a forcing term $\eta$, assumed to be a white in time Gaussian noise with autocorrelation function $E[\eta(r,t)\eta(r',t')] = C(r,r')\delta(t-t')$. $C$ is an even positive definite function, normalised such that the average energy input rate in (1) is $\alpha$. This model is known to give very good results for mid-latitude geophysical flows, and when $h = 0$, it reduces to the 2D-Navier-Stokes equations.

Equation (1) is written in non-dimensional units, such that the domain is $D = [0, 2\pi l_x] \times [0, 2\pi]$ (with aspect ration $l_x < 1$) and such that the approximate average energy, computed in a stationary state and neglecting the dissipation by viscosity, is 1. We note that in these non-dimensional units, $\alpha$ represents an inverse Reynolds’ number based on the large scale dissipation of energy and $\nu$ is an inverse Reynolds’ number based on the viscosity or viscosity term that acts predominantly at small scales. In the case of the beta-plane equations, the non-dimensional $\beta$ is the (square of the) ratio of the size of the domain over the Rhines scale [9].
2 Stochastic averaging

2.1 Rescaled dynamics

For the barotropic equations (1), the regime corresponding to the emergence of large-scale jets is given by $\alpha \ll 1$ (dynamical time scale of the jets much larger than the time scale of the turbulent fluctuations) and $\nu \ll \alpha$ (turbulent regime). This is the regime we consider in the following.

The large scale zonal jets are characterized by either a zonal velocity field $v_z(r) = U(y)e_x$ or its corresponding zonal potential vorticity $q_z(y) = -U'(y) + h(y)$. For reasons that will become clear in the following discussion (we will explain that this is a natural hypothesis and prove that it is self-consistent in the limit $\alpha \ll 1$), the meridional perturbation to this zonal velocity field is of order $\sqrt{\alpha}$.

We then have the decomposition

\[ q(r) = q_z(y) + \sqrt{\alpha} \omega_m(r), \quad v(r) = U(y)e_x + \sqrt{\alpha}v_m(r) \]

where the zonal projection is defined by $\langle f \rangle (y) = \frac{1}{2\pi l_x} \int_0^{2\pi l_x} dr \langle f \rangle$.

We now project the barotropic equation (1) into zonal and meridional part, assuming for simplicity that the random forcing doesn’t act directly on the zonal degrees of freedom $1$ ($\langle C \rangle = 0$):

\[ \partial q_z \partial t = -\alpha \partial \langle v_m,y \omega_m \rangle - \alpha \omega_z + \nu \partial^2 \omega_z \partial y^2, \]

\[ \partial \omega_m \partial t + L_U [\omega_m] = \sqrt{2} \eta - \sqrt{\alpha} \nabla v_m \cdot \nabla \omega_m + \sqrt{\alpha} \langle v_m, \nabla \omega_m \rangle, \]

where $L_U$ is the linearized dynamics operator around the zonal base flow $U$. We see that the zonal potential vorticity is coupled to the non-zonal one through the zonal average of the advection term \( \partial \langle v_m, y \omega_m \rangle \). If our rescaling of the equations is correct, we clearly see that the natural time scale for the evolution of the zonal flow is $1/\alpha$. By contrast, the natural time scale for the evolution of the meridional perturbation is one. These remarks show that in the limit $\alpha \ll 1$, we have a time scale separation between the slow zonal evolution and a rapid meridional evolution. Our aim is to use this remark in order to describe precisely the stochastic behavior of the Reynolds stress in this limit (by integrating out the meridional turbulence), and to prove that our rescaling of the equations and this time scale separation hypothesis is a self-consistent hypothesis.

2.2 Adiabatic elimination of fast variables

We will use the remarks that we have a time scale separation between zonal and meridional degrees of freedom in order to average out the effect of the meridional turbulence. This amounts at treating the zonal degrees of freedom adiabatically. This kind of problems are described in the theoretical physics literature as adiabatic elimination of fast variables [6] or may also be called stochastic averaging in the mathematics literature. Our aim is to perform the stochastic averaging of the barotropic flow equation and to find the equation that describes the slow evolution of zonal flows. In this stochastic problem, it is natural to work at the level of the probability density function (PDF) of the flow, $P[q] = P[q_z, \omega_m]$.

Complete Fokker-Planck equation

The evolution equation for the PDF reads

\[ \frac{\partial P}{\partial t} = \mathcal{L}_0 P + \sqrt{\alpha} \mathcal{L}_n P + \alpha \mathcal{L}_z P, \]

The operator $\mathcal{L}_0$ is the Fokker-Planck operator that corresponds to the linearized dynamics close to the zonal flow $U$, forced by a Gaussian noise, white in time and with spatial correlations $C$. This

1. This assumption is not necessary for the theory, it is just for convenience.
Effective Fokker-Planck equation acts on the meridional variables only and depends parametrically on $U$. This is in accordance with the fact that on time scales of order 1, the zonal flow does not evolve and only the meridional degrees of freedom evolve significantly. It should also be remarked that this term contains dissipation terms of order $\alpha$ and $\nu$. These dissipation terms can be included in $\mathcal{L}_0$ because in the limit $\nu \ll \alpha \ll 1$, the meridional dynamics is dominated by the interaction with the mean flow, thanks to the so-called Orr mechanism. This crucial point will be discussed in the following paragraph. At order $\sqrt{\alpha}$, the nonlinear part of the perturbation $\mathcal{L}_\nu$ describes the non-linear interactions between meridional degrees of freedom. At order $\alpha$, the zonal part of the perturbation $\mathcal{L}_z$ contains the terms that describe the large-scale friction and the coupling between the zonal and meridional flow.

**Stationary distribution of the fast variables**

The goal of our approach is to get an equation that describes only the zonal, slowly evolving part of the PDF, but taking into account the fact that the non-zonal degrees of freedom have rapidly relaxed to their stationary distribution. The first step is then to determine this stationary distribution of the meridional, fastly evolving degrees of freedom. This stationary distribution is given by the stationary state of (4), retaining only the first order term : $\mathcal{L}_0 P = 0$. For the special case of a determined zonal flow $P[q_z, \omega_m] = \delta(q_z - q_0) Q(\omega_m)$, $\mathcal{L}_0$ is the Fokker-Planck operator that corresponds to the dynamics of the meridional degrees of freedoms, for quasi-geostrophic equations linearized around the base flow with potential vorticity $\omega_0$ (equation (3) without the non-linear terms.) It is a linear stochastic process (Orstein-Uhlenbeck process) with zero average value, so we know that its stationary distribution is a centered Gaussian, entirely determined by the variance of $\omega_m$. The variance is the stationary value of the two-points correlation function of $\omega_m$, $g(r_1, r_2, t) = \mathbb{E}[\omega_m(r_1, t)\omega_m(r_2, t)]$. The evolution of $g$ is given by the so-called Lyapunov equation, which is obtained by applying the Ito formula to the stochastic equation for $\omega_m$:

$$
\frac{\partial \omega_m}{\partial t} + \mathcal{L}_U[\omega_m] = \sqrt{2\eta} \quad \Rightarrow \quad \frac{\partial g}{\partial t} + \left(L_U^{(1)} + L_U^{(2)}\right) g = 2C. \tag{5}
$$

($L_U^{(i)}$ means that the operator is applied to the $i$-th variable). We now understand that the asymptotic behaviour of this equation is a crucial point for the whole theory. It can be proved [2, 3] that $g$ has a well-defined limit (in the distributional sense) for $t \to \infty$, even in the absence of any dissipation mechanism ($\alpha = \nu = 0$). This may seem paradoxical as we deal with a linearized dynamics with a stochastic force and no dissipation mechanism. This is due to the Orr mechanism [2] (the effect of the shear through a non-normal linearized dynamics), that acts as an effective dissipation. The fact that (5) has a finite limit when $\alpha \to 0$ is the precise justification of the scaling (2.1), and it is thus the central point of the theory.

The average of an observable $A[q_z, \omega_m]$ over the stationary gaussian distribution is still a function of $q_z$, and it is an average over the non-zonal degrees of freedom, taking into account the fact that they have relaxed to their stationary distribution. In the following, we denote this average

$$
\mathbb{E}_U[A] = \int \mathcal{D}[\omega_m] A[q_z, \omega_m] G[q_z, \omega_m], \tag{6}
$$

the subscript $U$ recalling that this quantity depends on the zonal flow. With this definition, the adiabatic reduction can be performed. The details of the computation, that follow [6], are reported in [3]. Only the final result and its consequences are presented here.

**Effective zonal Fokker-Planck equation**

The final Fokker-Planck equation for the slowly evolving part of the zonal jets PDF $R[q_z]$ reads :

$$
\frac{\partial R}{\partial \tau} = \int dy_1 \frac{\delta}{\delta q_z(y_1)} \left( \frac{\partial F[U]}{\partial y_1} + \omega_z(y_1) - \nu \frac{\partial^2 \omega_z}{\partial y_1^2} \right) R[q_z] + \int dy_2 \frac{\delta}{\delta q_z(y_2)} \left( C R(y_1, y_2) [q_z] R[q_z] \right), \tag{7}
$$
which evolves over the time scale $\tau = \alpha t$, with the drift term

$$F[U] = E_U \langle v_{m,y} \omega_m \rangle (y_1) + \alpha F_1 [U],$$

with $F_1$ a functional of $q_z$, and the diffusion coefficient $C_R(y_1, y_2) [q_z]$, that also depends on the zonal flow $q_z$.

This Fokker-Planck equation is equivalent to a non-linear stochastic partial differential equation for the potential vorticity $q_z$,

$$\partial q_z / \partial \tau = - \partial F / \partial y \langle U \rangle - \omega_z (y_1) + \nu \Delta \omega_z + \zeta,$$

(8)

where $\zeta$ is white in time Gaussian noise with spatial correlation $C_R$. As $C_R$ depends itself on the velocity field $U$, this is a non-linear noise. The main physical consequences of this equation are discussed in the following paragraphs.

3 Physical interpretation of the zonal Fokker-Planck equation

3.1 First order : quasi-linear dynamics

At first order in $\alpha$, we obtain a deterministic evolution equation for $q_z$ :

$$\frac{\partial q_z}{\partial t} = -\alpha \frac{\partial}{\partial y} E_U \langle v_{m,y} \omega_m \rangle - \alpha \omega_z + \nu \Delta \omega_z,$$

(9)

where the forcing term $-\alpha \frac{\partial}{\partial y} E_U \langle v_{m,y} \omega_m \rangle$ can be computed as a linear transform of the stationary solution of the Lyapunov equation (5).

To summarize, we found that at leading order in $\alpha$, the zonal flow is forced by the average of the advection term due to the non-zonal fluctuations (Reynolds’ stress), and that this quantity is computed from the linearized dynamics for the fluctuations. In other words, we could have applied the same stochastic reduction technique to the quasi-linear dynamics (equations (2) and (3) without non-linear terms), and we would have obtained at leading order the same deterministic equation (9). The system (9,5) is a quasi-Gaussian (or second-order) closure of the dynamics. Working directly at the level of the PDF, and using the tools of the stochastic reduction, we have been able to justify the closure of this problem. This quasi-Gaussian closure has been already studied in numerical works (SSST in [5] and CE2 in [8]), and is known to give very good results.

Using again the results about the Orr mechanism [2, 3], some important facts about equation (9) can be proved. First, we can make sure that the Reynolds’ stress is well-behaved, even in the inertial limit $\alpha, \nu \rightarrow 0$, so that the zonal flow equation (9) is always well-defined. We can also show that the energy in the meridional degrees of freedom is of order $\alpha$. As a consequence, a vanishing amount of energy is dissipated in the fluctuations and almost all the energy injected by the stochastic forcing goes to the zonal degrees flow. Moreover, the dynamics defined by equations (9,5) are much more simpler to solve numerically that the full non-linear dynamics (1). The numerical results that illustrate the two properties mentionned above are reported in figure 2.

3.2 Next order : corrections and multistability

From the full Fokker-Planck equation (4), we expect the non-linear operator $L_\alpha$ to produce terms of order $\alpha^{1/2}$ and $\alpha^{3/2}$ in the zonal Fokker-Planck equation (7). The detailed computation shows that these terms exactly vanish. As a consequence, we have proved that the quasi-Gaussian closure (9,5) is correct in the limit $\alpha \ll 1$, with correction only at order $\alpha^2$.

We then have a correction $F_1$ to the drift $F[U]$ due to the non-linear interactions. At this order, the quasilinear dynamics and non-linear dynamics differ. We also see the appearance of the noise term, which has a qualitatively different effect than the drift term. For instance if one is interested in large deviations from the most probable states, correction of order $\alpha$ to $F_0$ will still be vanishingly small, whereas the effect of the noise will be essential. This issue is important for the description of the bistability of zonal jets and phase transitions.
4 Conclusion

We have shown that it is possible to average out the effect of turbulence in the problem of the jet formation in the barotropic model, when there is a time scale separation between the evolution of the jets and the turbulent dynamics. Instead of following the classical route, based on an arbitrary closure in the Reynolds’ hierarchy of equations, we performed this closure working directly at the level of the probability distribution function of the vorticity field. The main aspects of the equation we obtain are the following: at first order, it describes the quasi-Gaussian closure, and thus justifies theoretically the previous studies on the subject. Then, it predicts the corrections to this approximation, and allows the description of the bistability of the jets. This last point is the subject of ongoing research.

Références

[1] F. Ait Chaalal, private communication