Theoretical and numerical studies on the elastic-plastic buckling risk of tools and work-pieces during forging processes

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Abstract:
The main goal of this study is to validate theoretical and numerical models used to analyse the elastic, elastic-plastic and purely plastic buckling. It is known that in the materials bulk forming industry some tools (supports, punches or ejectors) presents a risk of buckling, especially in the field of elastic deformations. However, the work pieces undergoing large plastic deformations during forging operations and can have sometimes a significant risk of plastic buckling. Currently there are graphs or diagrams defining the ratios to avoid this buckling risk, but the limits of use are not known and the extrapolations are very difficult to be realized. For a more rigorous analysis is proposed to use finite element simulations. To estimate the critical buckling load the variation curves of the longitudinal force and of the lateral displacement according to the vertical movement will be used, identifying the critical values of the force characterizing the elastic or plastic instability by the point responsible for the observed break slope of the previous curves.

Résumé:
L’objectif de cette étude est de valider des modèle théoriques et numériques utilisés pour l’analyse du phénomène de flambage élastique, élasto-plastique et purement plastique. Il est connu dans l’industrie de formage volumique des matériaux que certains outillages (appuis, poinçons ou éjecteurs) présentent des risques de flambage en service, surtout dans le domaine des déformations élastiques. Cependant les pièces subissent lors des opérations de forgeage des grandes déformations plastiques et présentent parfois un risque important de flambement. Actuellement il existe des abaques définissant les ratios à respecter pour éviter ce risque, mais les limites d’utilisation ne sont plus connues et les extrapolations sont difficiles à être réalisées. Pour obtenir une analyse plus rigoureuse on propose l’utilisation des simulations éléments finis. Ainsi pour l’estimation de la charge critique de flambage on utilisera les courbes de variation de la force longitudinale et du déplacement transversal en fonction du déplacement vertical, en identifiant le point critique de sollicitation caractérisant l’instabilité de flambement par le point responsable de la rupture de pente observée.

Key Words: elastic/plastic buckling analysis, theoretical models, finite elements modelling

1 Introduction
The industrial manufacturing of metallic materials remains to nowadays often conditioned by a bulk forming process: forging, extrusion, stamping, spinning… Some forging tools, such as supports, punches or ejectors can have important risk of buckling during the use (Fig. 1). The critical loads of stability can be determined by using methods based on the mechanical equilibrium equation for a state slightly perturbed around the initial position [1], taking into account the energy methods [2,3] or by the consideration of some imperfections [4]. Hence, the theoretical analysis of the beams buckling use the study carried out by Euler from an analytical model based on an elastic behaviour defined by a Hooke law. This theory shows that the elastic buckling can be studied by using the curve of the critical stress variation with a defined slenderness factor. In a similar way, several empirical models as Tetmayer-Jasinski’s or Johnson’s ones are proposed in the literature [4-6] to characterize experimentally the elastic-plastic and plastic buckling. This paper proposes a new analytical model to characterize the elastic-plastic buckling taking into account a linear variation of the critical stress defined by a tangent fitting with the elastic Euler hyperbola. This model will be validated by comparisons with the experimental results shown in the literature and especially by numerical simulations using the FE codes Cast3M and Abaqus/Standard. The proposed numerical simulations will be analyzed to validate their feasibility in the description of the buckling process corresponding to the elastic-plastic and plastic domains.
2 Elastic buckling

2.1 Theoretical analysis

In the field of materials strength, the buckling is the tendency of a compression-stressed beam to flex and therefore to deform in a direction perpendicular to the applied force. One of the basic assumptions in the elastic computation of beams or structures is to assume small displacements and small deformations. Otherwise, the equilibrium conditions must be written with respect to the deformed configuration of the structure when it is even possible that the corresponding deformations become very important. For slender structures, there are also instability phenomena, i.e. that for a given load there is no more uniqueness of the deformation. Classically the theoretical analysis of buckling of a compressed beam is based on the study developed by Euler. From this theory where a thin rectilinear rod is subjected to a vertical force $P$ lower than a critical value $P_{cr}$, a single position of stable equilibrium exists, that in which the rod remains straight. When $P$ is greater than the critical load, two equilibrium positions exist: one in which the rod is bent, other in which the shaft remains straight but very unstable (Table 1).

### TABLE 1 – Euler elastic buckling bending of a beam using different boundary conditions

<table>
<thead>
<tr>
<th>Buckling Load Cases</th>
<th>Characteristic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{cr} = \frac{\alpha^{2}EI_{min}}{l}$</td>
<td>$\sin(\alpha l) = 0$</td>
</tr>
<tr>
<td>$d^{2}w}{dz^{2}} + \alpha^{2}w = 0$</td>
<td>$cos(\alpha l) = 0$</td>
</tr>
<tr>
<td>I, I' ($l_{j} = l$)</td>
<td>$tg(\alpha l) - \alpha = 0$</td>
</tr>
<tr>
<td>II, II' ($l_{j} = 2l$)</td>
<td>$cos(\alpha l) - 1 = 0$</td>
</tr>
<tr>
<td>III ($l_{j} = 0.7l$)</td>
<td></td>
</tr>
<tr>
<td>IV ($l_{j} = 0.5l$)</td>
<td></td>
</tr>
</tbody>
</table>

The main assumptions used by the Euler analysis assume: • a perfect homogeneous and isotropic material considering to have an ideal linear elastic behaviour (plastic strain due to the compressive force is then neglected); • a constant section of the beams with a perfect straightness of their mean line and rigorously articulated at their ends; • a plane strain bending of the bars about a principal inertia axis of cross section. Introducing the concept of the slenderness factor $\lambda = \frac{l_{j}}{i}$, where $i$ is the gyration radius defined by $i = \sqrt{I_{min}/S}$ ($S$ is the section area and $I_{min}$ is the minimum moment of inertia of the cross section), the Euler critical load can be computed by $P_{cr} = \frac{\pi^{2}EI_{min}}{l_{j}^{2}}$ or $P_{cr} = \frac{\pi^{2}ES}{\lambda^{2}}$ where the uniform Euler critical stress is defined by $\sigma_{cr} = P_{cr}/S = \frac{\pi^{2}E}{\lambda^{2}}$ (Euler hyperbola). Starting from Euler equation and for the material’s elastic yield strength $\sigma_{p}$ we have $\sigma_{cr} = \sigma_{p} = \frac{\pi^{2}E}{\lambda_{0}^{2}} \Rightarrow \lambda_{0} = \frac{\pi}{\sqrt{E/\sigma_{p}}}$. Thus, for high values of slenderness, if $\lambda \in [\lambda_{0}, \lambda_{max}]$, the Euler formula can be used for the calculation of the elastic buckling, where $\sigma_{cr} \leq \sigma_{p}$. The coefficient $\lambda_{0}$ depends only on the beam material and the maximal values $\lambda_{max}$ are generally determined from technical standards.

2.2 Numerical analysis

It is considered a cylindrical bar, supposed isotropic and homogeneous, with the diameter $D = 10$ mm, the length $l = 150$ mm and subjected to a vertical compression corresponding to the load case IV (Table 1).
slenderness factor is here \( \lambda = l_f / l = 0.5l / l = 30 \) and the general mechanical characteristics are defined by the mass density \( \rho = 7.85 \times 10^9 \) tones / mm\(^3\), the modulus of Young \( E = 200000 \) MPa and the coefficient of Poisson \( \nu = 0.3 \). FE simulations were performed with the Cast3M code (using a specific procedure for elastic buckling, giving the critical load through a multiplier coefficient) and with ABAQUS/Standard software (using a classical 3D FE model of a static compression choosing displacement increments according to a constant speed \( V = -1 \) mm/\( s \)). The FE mesh uses linear cubic elements with 8 nodes and with a size of 2 mm (total number of elements 2088). The evolution of the cylindrical beam deformation is presented in the Fig. 2a.

\[
\sigma = \frac{P}{A} = \frac{\pi d^2}{4} \frac{F}{l} = \frac{E}{\lambda^2} \lambda \sigma_0
\]

To determine the critical load of buckling, we proceed in the following way: · First stage: the displacements of all points of each element are expressed in a unique way according to the nodes displacements. As the upper surface is fixed, in order to determine the movement in Y direction, we chose the node placed in this direction, at 2 mm of the upper surface, of coordinates: 3.535 mm, 3.535 mm, 148 mm. We draw the graph of variation of the horizontal movement of this node following the axis OY according to the vertical movement following the axis OX (Fig. 2b). We examine the graph and we notice that the buckling occurs in the point A (displacement of 1.5 mm on OX axis and displacement of 0.024 mm on OY axis). It is observed that starting from this point, the horizontal displacement continue to grow in a significative way with a very important change of the slope. · The second stage: we draw the evolution of the load according to the vertical movement of the same node and we identify the value corresponding to a 1.5 mm of movement (Figure 2b). The value of the strength in the point B can be taken into account as being the critical load for elastique buckling. Finally, we obtain \( P_{cr}^{EF} = 157300 \) N and a critical stress \( \sigma_{cr}^{EF} = P_{cr}^{EF} / S = 2002.8 \) MPa. The error compared with the theoretical Euler critical stress estimated by 2193.2 MPa is about 8.682 %. Using the Cast3M code \( P_{cr}^{lim} = 172257 \) N and \( \sigma_{cr}^{lim} = 2252.482 \) MPa i.e. a relative error of 2.7 % compared with the theoretical values. Similar good accuracy (maximum of 5%-8%) is obtained for square or triangular section shape of the beam.

### 3 Elastic-plastic and plastic buckling

The real elements of a structure do not behave exactly as the predictions of the elastic bifurcation theory. First of all, the material is not infinitely elastic; in reality elastic-plastic behaviour occurs during the deformation process and thus the piece rather risks of plastic buckling.

#### 3.1 Theoretical and empirical models

From the Euler theory and some experimental results, various analytical relations can be established to define variation of \( \sigma_{cr}^e(\lambda) \) outside of elastic domain (Fig. 3). In this way, for short bars ( \( \lambda \leq \lambda_0 \) and \( \sigma_p < \sigma_{cr} < \pi^2 E / \lambda^2 \)), if \( \lambda \in [\lambda_f, \lambda_0] \), by using a real elastic-plastic behavior, taking into account the hardening effect defined by \( \sigma \in [\sigma_p, \sigma_c] \), the buckling is instead in an elastic-plastic domain. But if \( \lambda < \lambda_f \), the behavior becomes a perfectly plastic one i.e. \( \sigma = \sigma_c \) and we can consider that the material is in the purely plastic domain where the buckling can be estimated from a pure compression i.e. \( \sigma_{cr} = \sigma_c \).
3.1.1 Tetmayer-Jasinski model and the proposed improvement

For an elastic-plastic buckling computation \((\lambda \in [\lambda_1, \lambda_0])\), the most used empirical model is the Tetmayer-Jasinski one \([4,6]\) defined by
\[
\lambda \sigma_{ba}^{cr} - \lambda = (a - b \lambda),
\]
where \(a, b\) are constants which depend on the beam’s material.

The scientific literature \([6]\) gives different values, \(a^e, b^e, \lambda^{e}_{a_0}\) and \(\lambda^{e}_{1}\), obtained experimentally for a class of materials. An analysis of this model shows that normally \(\lambda_0 = \pi_0 \sqrt{E/\sigma_p} = (a - \sigma_p)/b\) and \(\lambda_1 = (a - \sigma_c)/b\).

But the values of \(\lambda_1\) and \(\sigma_c\) are not a priori known and an estimation of coefficients \(a\) and \(b\) by analytical estimation becomes impossible. However Eurocodes standards \([7,8]\) indicates the choice mainly for \(\lambda_1 \approx 0.2 \lambda_0\). To improve this empirical model a new formulation is proposed, considering a tangent line to the Euler hyperbola in the point \((\sigma_p, \lambda_0)\), defined by \(\sigma_{cr} = a' - b' \lambda\). Thus for \(\lambda = \lambda_0 \Rightarrow \sigma_p = a' - b' \lambda_0\) and by adding the property of the derivative’s continuity in \(\lambda = \lambda_0 \Rightarrow b' = -d\sigma_{cr}/d\lambda = 2 \cdot \pi^2 \cdot E/\lambda_0^2\) and \(a' = 3\sigma_p\).

To define completely this new formulation the \(\sigma_c\) can be computed from the equation \(\lambda_1 = (a' - \sigma_c)/b' = 0.2 \lambda_0\). Analyzing the data obtained for different steels \([6]\), by comparing the analytical and experimental values of the coefficients \(a', b', a, b\) the following formula can be used: \(a = a'/2, b = b'/(2.5 \div 3.5), \lambda^{e}_{a_0} = 1.5 \lambda_0\) and \(\lambda^{e}_{1} = 4 \lambda_1\). So for the S355 steel (\(\sigma_p = 355\) MPa) the coefficients are defined by: \(\lambda_0 = 74.568, \lambda_1 = 15.914, a = 532.5\) MPa, \(b = 3.174\) MPa, \(\sigma_c = 481.989\) MPa i.e. \(\sigma_{cr} = 532.5 - 3.174 \lambda\).

3.1.2 Johnson Model

Another relation used for the elastic-plastic buckling domain is the Johnson formula \([6]\), represented by a parabolic function \(\sigma_{cr} = a'' - b'' \lambda^2\) connected with the Euler hyperbola in the point \((\sigma_p, \lambda_0)\). The values of \(a''\) and \(b''\) can be determined in the following way: for \(\lambda = 0 \Rightarrow a'' = \sigma_{cr} = \sigma_c\); for \(\lambda = \lambda_0 = \pi \sqrt{E/\sigma_p}\), which is the junction point with the Euler curve \((\sigma_{cr} = \sigma_p)\), using the continuity of the derivative of Johnson’s parabola and of the Euler hyperbola, we obtain \(b'' = \sigma_p^2 / \pi^2 E\). Thus in the elastic-plastic domain, the critical stress defined by the Johnson model becomes \(\sigma_{cr} = \sigma_c - (\sigma_p^2 / \pi^2 E) \lambda^2\). The majorities of applications use a stress flow \(\sigma_c\) defined by \(\sigma_c = 2 \sigma_p\). Compared with the relation of Tetmayer-Jasinski in this case are defined only two domains: the elastic zone (Euler formula) and, because \(\lambda_1 = 0\), the elastic-plastic one (Johnson formula). If we retake the experimental results obtained for the S355 steel we obtain \(\sigma_{cr} = 710 - 0.064 \lambda^2\).

3.2 Finite element analysis

Firstly, it is analyzed the plastic buckling of a perfectly rectilinear compressed cylinder of a S355 steel defined by a perfect elastic-plastic behaviour law i.e. \(\sigma = \sigma_p = 355\) MPa. This rigid-plastic behavior supposes that we have neither lateral nor axial plastic deformation mainly for low values of the normal load \(P\). The analysis is
performed by means of Abaqus/Standard software using for the specimen the same geometrical characteristics and the same boundary conditions as we have taken into account in the purely elastic study. To determine the critical strength and stress of the buckling phenomenon, we proceed in the same way as in the case of the elastic one. The Fig. 4a pictures the obtained evolution of the cylindrical bar deformation.

![Graphs and diagrams showing deformation evolution.]

**FIG. 4**  a) Numerical results obtained from Abaqus/Standard concerning the evolution of the buckling mode corresponding to a rigid elastic-plastic behaviour, b) Evolution of the displacement and load for the point situated on the surface of the bar at 2 mm distance from the upper section

If we draw the graph of horizontal movement of the node situated at 2 mm distance of the upper section following the axis OY, according to the vertical movement following the axis OZ (Fig. 4b), we notice that the buckling occurs at a time of 0.3s in the point A (0.3 mm, 0.002 mm). For this point (situated at a 0.3 mm of the displacement on the axis Oz) it is observed that a small displacement of 0.002 mm occurs on the axis Oy, after that the continuous horizontal movement increase with a greater slope. The evolution of the load according to the vertical displacement of the same node permits to identify the corresponding value at the displacement of 0.3 mm (point B). The value of the load in this point represents thus the critical strength of the plastic buckling. The obtained numerical values are $P_{pl}^{EF} = 27200$ N and $\sigma_{cr}^{EF} = P_{pl}^{EF} / S = 346.32$ MPa. Using an analytical model the slenderness is defined by $\lambda = 30$ and $\lambda_0 = \pi \sqrt{E / \sigma_p} = 74.568$. So $\lambda \leq \lambda_0$ and the theoretical critical stress is $\sigma_{cr}^{th} = \sigma_p = 355$ MPa (Fig. 3a). The corresponding empirical critical plastic load is

$$P_{pl}^{th} = \sigma_{cr}^{th} S = 27881 \text{ N, consequently the error as compared to the numerical value is } \left| \frac{\sigma_{cr}^{EF} - \sigma_{cr}^{th}}{\sigma_{cr}^{th}} \right| = 2.44 \% ,$$

less than 3%. This comparison permits to validate the numerical model and especially the use of the FE modelling to analyze plastic buckling phenomenon. According to a real elastic-plastic behaviour of the material (S355), it is necessary to introduce a hardening law defined by $\bar{\sigma} = \sigma_p + B \bar{\varepsilon}^n$ with $\sigma_p = 355$ MPa, $B = 500$ MPa and $n = 0.5$. Normally, the cumulated plastic strain $\bar{\varepsilon}$ influences the material plastic flow and contributes to an increase of the collapse load as compared to the previous case, because the plastic deformations prevail and should raise the corresponding zone of the buckling strength curve as compared with the rigid elastic-plastic part. Furthermore this effect appears for low values of the load as those of a purely elastic behaviour. The evolution of the cylindrical bar deformation obtained by FE simulation is presented in the Fig. 5a. To estimate the critical buckling strength and the corresponding critical stress, we proceed in the same way as in the previous cases (Fig. 5b). The critical load is identified at a 5 mm of the vertical displacement for which we observe a rough change of the displacement curve slope. The value of the strength in this point represents the critical load of the plastic buckling ($t = 5s$) i.e. $P_{pl}^{EF} = 33500$ N and the corresponding critical stress is $\sigma_{cr}^{EF} = P_{pl}^{EF} / S = 426.90$ MPa. Using the proposed modified Tetmaier-Jasinski model, the empirical critical plastic strength is $P_{pl}^{th} = \sigma_{cr}^{th} \cdot S = 34343.9$ N with a theoretical critical stress $\sigma_{cr}^{th} = 532.5 - 3.174 \cdot 30 = 437.28$ MPa; in this case the compressed bars have a behaviour defined in the elastic-plastic domain described by $15.914 = \lambda_1 < \lambda < \lambda_0 = 74.568$. The error $\left| \frac{\sigma_{cr}^{EF} - \sigma_{cr}^{th}}{\sigma_{cr}^{th}} \right|$ is equal to 2.4 %, value which confirms simultaneously the feasibility of FE results and those of the proposed empirical model. If we compare the numerical result with the value of the critical stress estimated using the Johnson
model \((\sigma_c = 710 - 0.064 \cdot 30^2 = 652.4 \text{ MPa})\), the error is around 34.6\%, greater than the previous one, caused by the overestimation of the flow stress: \(\sigma_c \equiv 2\sigma_p = 710 \text{ MPa}\).

FIG.5 – a) Numerical results obtained from Abaqus/Standard concerning the evolution of the buckling mode corresponding to a real elastic-plastic behaviour taking into account the hardening effect, b) Evolution of the displacement and load for the point situated on the surface of the bar at 2 mm distance from the upper section.

The experimental tensile tests of S355 steel show that \(\sigma_c\) is about 510 MPa and consequently \(\sigma_c = 510 - 0.064 \cdot 30^2 = 452.4 \text{ MPa}\) with an error as compared to the numerical value of 5.63\%.

4 Conclusions

The FE results concerning the elastic or plastic buckling during a simple compression of a cylindrical beam, embedded in both extremities, show the possibility to analyze numerically this instability phenomenon. By comparing the numerical values of the critical elastic strength obtained from Cast3M and those of the analytical Euler formulas, it is possible to conclude that this software gives the results with a high accuracy.

Using a standard version of the Abaqus software, starting only from definition of boundaries conditions and of the material behaviour, using a classical FE analysis and a particular strategy to identify the critical point of the specimen instability, coherent results were also obtained. Furthermore the FE modelling show that the plastic buckling can be analyzed in conditions close to the real material behaviour and that all the numerical results obtained for rigid elastic-plastic behaviour or for a real elastic-plastic one are in good agreement with the empirical models, especially for a class of steels. In a more general way, the use of the FE modelling taking into account the real geometry, load patch, boundary conditions and real behaviour law, allows identifying the mechanical condition of the elastic or plastic buckling risk for tools or work pieces with complex shapes. In these cases, any main criterion able to estimate the decrease of the buckling risk supposes especially that the applied load should be lower than the critical value obtained by numerical simulations.

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