Reconstruction of impact force on elastic structures by using Bayesian approach

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Abstract:
In this work, reconstruction of pressure signal developed during non-punctual impact occurring on an elastic beam has been achieved through using Bayesian approach for inverse problem. This was performed by means of posterior probability density function that takes into account both the likelihood and prior random information. A new direct method was proposed to achieve pressure signal reconstruction when the system model is assumed to be linear with additive noise and prior and noise distributions of probabilities are taken to be independent Gaussians with proportional diagonal covariance matrices. It was found that reconstruction works very well in this situation as it was very rapid and the obtained pressure was very close to the original pressure taken at the input of the system.

Key words: inverse problem, Bayesian approach, impact, elastic structure, force reconstruction, Wiener filter.

1 Introduction
Knowledge of forces exerted on a mechanical system constitutes vital information in various applications dealing with structural health monitoring. However, direct measurement of the force by transducers is so difficult in practice, and even impossible in many situations. This is why indirect methods are preferred in this field. These consist in reconstruction of the force by solution of an inverse problem. Various methods have been developed during last decades which enable identification of force characteristics through measuring the response in some locations of the elastic structure [1]. Considering the time domain approach to the problem, great attention has been paid to the stability issue of inversion operation like for instance applying singular value decomposition followed by Tikhonov regularisation [2].

Force reconstruction has also been considered within the context of Bayesian approach which assumes the unknowns as random variables [3,4]. This approach provides a rigorous probabilistic framework that has the capacity of incorporating all sources of uncertainties including those resulting from the actual model used or those associated to noise affecting measurement. The Bayesian based approach takes benefit from both the prior available information and the posterior information provided by the measured response in order to effect regularization of inversion. Estimation of the inverse problem solution can then be obtained by the
conditional mean of the posterior probability density function. Bayesian reconstruction of the force is conducted generally through Markov Chain Monte Carlo sampling method (MCMC) [5,6].

The Bayesian approach for force reconstruction in the frequency domain uses the Frequency Response Function (FRF) of the considered system and takes advantage from the deconvolution theorem which states it as a simple division in the frequency domain. However the division in the frequency domain can only be accurately computed as long as the frequency is far from the structural resonances. Otherwise, the obtained results could be very affected by the measurement noise. They depend also largely on the actual damping present in the system, so highly accurate estimation of damping is required when crossing the FRF singularities. Moreover, spectral analysis based methods use the Fourier transform and its inverse in order to get the data in the frequency domain and to transform it back into the time domain. This operation is generally accompanied by some filtering noise and imposes restrictions on data sampling.

In this work, a new regularization technique is proposed. It is based on a two step iteration algorithm which performs remarkably regularisation of the Toeplitz like matrix inversion and enables computing the covariance matrix of the posterior distribution. Given the measurement, this method was shown to provide the priori and posterior means with excellent accuracy.

This methodology will be applied to reconstruct the impact pressure developed during a non punctual impact occurring on an elastic beam.

2 Direct problem of impact with uniform pressure on an elastic beam

A planar beam having a uniform section and loaded orthogonally to its mean fiber in its plane of symmetry is considered. It is assumed to be made from a homogeneous elastic material and having a rectangular uniform cross section, figure 1.

The beam is assumed to be subjected to a force resulting from a non punctual impact where the acting pressure is taken uniform on the whole impact zone, considered to be perfectly known. The inverse problem is then only about reconstruction of the pressure time signal from measured strains on some beam locations. The impact pressure is of the form \( p(x_0, t) \) where \( x_0 \) designates the center of the impact zone, \( \eta \) its longitudinal extension \( [x_0 - \eta/2, x_0 + \eta/2] \) and \( t \) the time. The measured longitudinal normal strain \( y(x_c, t) \) corresponds to a measurement point of abscissa \( x_c \) located in the upper beam fiber.

Let \( p \) and \( y \) denote respectively the vector containing the unknown discrete values of the pressure and measured strain, the Toeplitz like matrix \( G \) giving the discrete time response, in terms of the longitudinal axial strain of the upper fiber at position \( x_c \), as function of the discrete vector \( P \) representing the pressure of impact, writes

\[
G(k, j) = \frac{2\pi \sin(\omega x_0)}{\rho S L^2} \sum_{m=1}^{M} \sin\left(\frac{m\pi x_0}{L}\right) \sin\left(\frac{m\pi \eta}{L}\right) \sin\left(\frac{m\pi \eta}{2L}\right) g(\omega_m, \xi_m, (j-k)\Delta T)
\]

with \( g(\omega_m, \xi_m, (j-k)\Delta T) = 0 \) if \( j < k \), and

\[
g(\omega_m, \xi_m, (j-k)\Delta T) = \frac{\exp(-\xi_m(\omega_m(j-k)\Delta T))}{\omega_m \sqrt{1-\xi_m^2}} \sin\left(\omega_m \sqrt{1-\xi_m^2}(j-k)\Delta T\right) \quad \text{if} \quad j \geq k
\]
and

\[ \omega_m = \frac{m^2 \pi^2}{L^2} \sqrt{\frac{EI}{\rho S}} \]

(3)

where \( E \) is the Young’s modulus, \( \rho \) the density, \( I \) the moment of inertia, \( S \) the cross section area, \( L \) the beam length, \( h \) the height of the beam cross section, \( \xi_m \) the modal damping associated to mode number \( m \), \( \Delta t \) the time step and \( M \) the total number of modes that are retained due to modal truncation.

The applied pressure is assumed to be of the form \( p(t) = a t^2 e^{-bt} \). It reaches the maximum \( p_{\text{max}} = 4a/(b^2e^2) \) for \( t_{\text{max}} = 2/b \). Parameters \( a \) and \( b \) are linked to the pressure profile by \( a = p_{\text{max}}e^2/t_{\text{max}}^2 \) and \( b = 2/t_{\text{max}} \).

### 3 Direct Bayesian approach to pressure reconstruction

Assuming that pressure \( p \) and measurement \( y \) are random variables, the posterior density of probabilities is given according to Bayes by

\[ \pi(p/y) = \frac{\pi(y/p)p_{\text{pr}}(p)}{\pi(y)} \]  

(4)

where \( p_{\text{pr}}(p) \) is the prior density of probabilities rendering some available information about the unknown pressure \( p \), \( y/p \) is the likelihood density of probabilities giving the probability that observation be \( y \) when the pressure is \( p \) and \( \pi(y) \) is the density of probabilities of observations.

The beam model that integrates the presence of some additive measurement noise is considered under the following linear form

\[ Y = GP + E \]

(5)

where \( G \) is a random vector representing measurement noise. In the following \( E \) and \( P \) are assumed to be distributed according to multi-variable Gaussians: \( P \sim N(p_0, \Gamma_{\text{pr}}) \) and \( E \sim N(0, \Gamma_{\text{noise}}) \) where \( \Gamma_{\text{pr}} \) and \( \Gamma_{\text{noise}} \in \mathbb{R}^{n \times n} \) are symmetric positive definite matrices, thus invertible, while the expectation \( p_0 \) constitutes the unknown prior distribution mean. The posterior probability density of \( P \) given the measurement \( Y = y \) is then shown [7] to be obtained as

\[ \pi(p/y) \propto \exp \left( -\frac{1}{2}(p - \bar{p})^T \Gamma_{\text{post}}^{-1}(p - \bar{p}) \right) \]  

(6)

with

\[ \bar{p} = \Gamma_{\text{post}}^{-1} \left( G^T \Gamma_{\text{noise}}^{-1} y + \Gamma_{\text{pr}}^{-1} p_0 \right) \]  

(7)

and

\[ \Gamma_{\text{post}} = \left( \Gamma_{\text{pr}}^{-1} + G^T \Gamma_{\text{noise}}^{-1} G \right)^{-1} \]  

(8)

We consider the particular case where the covariance matrices \( \Gamma_{\text{pr}} \) and \( \Gamma_{\text{noise}} \) are proportional to identity \( \Gamma_{\text{pr}} = \sigma^2 I \) and \( \Gamma_{\text{noise}} = \nu^2 I \) and assumed to be known. In order to get estimation of the posterior mean \( \bar{p} \), we introduce the following iteration

\[ p_0 = \text{inv} \left( G'G + \epsilon I \right) G'y \]

(9)

\[ \bar{p} = \left[ \text{inv} \left( G'G + \frac{\nu^2}{\sigma^2} I \right) \right] \left( G'y + \frac{\nu^2}{\sigma^2} p_0 \right) \]  

(10)
where $\varepsilon$ is a scalar having a small value that is fixed to effect regularization of the Toeplitz matrix inversion and $\text{inv}$ is the classic Matlab command used to compute the inverse of a square matrix.

While equation (9) gives an estimation of the unknown prior mean $p_0$, equation (10) enables calculating the posterior mean and provides an estimation of the inverse problem solution. Equation (11) gives the variability that affect the reconstructed pressure mean $\bar{p}$ due to noise and for known prior standard deviation.

One should notice that two regularisations are successively performed in the algorithm defined by equations (9) to (11), in order to compute the centre point and the posterior covariance matrix. One can see that the second step constitutes a generalisation of the Wiener filter solution \[8,9\] of the deterministic ill-conditioned problem $Y = GP$. To see that let's assume that $P \sim N(0, \sigma^2 I)$ and $E \sim N(0, \nu^2 I)$, one obtains then easily by letting $p_0 = 0$ the following equation

$$\bar{p} = G' \left( GG' + \frac{\nu^2}{\sigma^2} I \right)^{-1} y = \left( G' G + \frac{\nu^2}{\sigma^2} I \right)^{-1} G' y$$

which is the well known regularized solution of system inversion $Y = GP$ by Wiener filtration.

The novelty here consists in estimating $p_0$ through equation (9). Parameter $\varepsilon$ was introduced to regularize inversion of the Toeplitz matrix $G$. This can for instance be taken to be $\varepsilon = \frac{\nu^2}{\sigma^2}$. So, knowledge of noise and prior variances are sufficient to get pressure reconstruction $\bar{p}$ even if the prior mean $p_0$ is unknown. The assumption regarding $\sigma$ and $\nu$ is not a big limitation since in practice these parameters can be estimated from experiment and simulation of potential impacts.

4 Case study

Let's consider a pinned-pinned beam as shown is figure1 for which Young's modulus is $E = 2.1 \times 10^7$ Pa and density is $\rho = 7800$ kg.m$^{-3}$. The beam length is $L = 1$ m, its width is $w = 2 \times 10^{-2}$ m and the height is $h = 5 \times 10^{-2}$ m. Damping is chosen to be $\xi = 5 \times 10^{-3}$ and impact characteristics are $x_0 = 5L/6$, $\eta = L/6$ and $x_s = L/3$.

The first 5 modes were retained, so that $M = 5$. The 5 first frequencies of the pinned-pinned beam are: $f_1 = 73.92$ Hz, $f_2 = 295.7$ Hz, $f_3 = 665.2$ Hz, $f_4 = 1183$ Hz and $f_5 = 1848$ Hz. The total duration of calculation was fixed at $T_c = 0.12$ s and the chosen time step value was $\Delta t = 2.1646 \times 10^{-4}$ s.

The Toeplitz matrix dimension which is equal to the length of the strain response $\mathbf{y}$ and the unknown vector $\mathbf{p}$ was then equal to 555. The pressure signal considered is that associated to the constants $a = 2 \times 10^7$ and $b = 500$.

Applying statistical inversion as defined by the algorithm defined by equations (9) to (11) enabled to estimate the conditional mean of pressure signal amplitude $\bar{p}$. Figure 2 presents the reconstructed and the original pressure signals. This figure shows that reconstruction of impact pressure achieved remarkable accuracy as the two signals are perfectly superposed.
5 Conclusion

Reconstruction of pressure developed during impact occurring on elastic beams has been achieved through using the Bayesian approach. This was performed through using the posterior distribution that integrates both the likelihood and prior information. Considering the case of linear noisy model where all the densities of probabilities are independent Gaussians with proportional covariance matrices, a direct method was proposed to reconstruct the pressure time profile and to evaluate variability affecting it due to noise and available prior information. The method is based on a new regularization scheme and consists on two obvious steps, limiting thus the computational effort needed with MCMC based methods. In the actual analysis the noise and prior covariance matrices were assumed to be known, it would be interesting to investigate how these can be identified from the system configuration and nature of impact that is likely to occur.

References