Numerical solution of the free convection induced by a heated vertical flat plate embedded in a saturated porous medium with internal heat source

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Résumé :

Dans ce travail, nous étudions la modélisation numérique du problème de la convection autour d'une plaque plane chauffée et enfoncée verticalement dans un milieu poreux saturé par un fluide Newtonien en présence d'une source de chaleur interne. L'écoulement hydrodynamique couplé avec le transfert de chaleur sont décrits par l'équation de continuité, l'équation de Darcy et l'équation de la chaleur. Ces équations sont transformées en forme similaire adimensionnelle pour obtenir une équation non linéaire fermée par des conditions aux limites. Le problème est résolu numériquement en utilisant la méthode de Runge-Kutta d'ordre 5 associée à une méthode itérative. Les résultats obtenus permettent d'illustrer les profils de vitesse et de température. Finalement, les effets des paramètres du problème sur les résultats obtenus sont discutés d’avantage.

Abstract :

In this work, we study the numerical modeling of the natural convection induced by a heated vertical flat plate embedded in a saturated porous medium by a Newtonian fluid in the presence of an internal heat source. The hydrodynamic flow coupled with heat transfer is described by the equations of continuity, Darcy’s and heat transfer. These equations are transformed into dimensionless form similar to obtain a nonlinear equation closed by the boundary conditions. The problem is solved numerically using a fifth order Runge-Kutta and iteration methods. The obtained results illustrate the profiles of velocity and temperature. Finally, the effects of the problem parameters on physical phenomena are discussed further.

Mots clés : Free convection ; Saturated porous medium ; Numerical solutions

1 Introduction

The study of heat transfer in a saturated porous medium is very important in recent decades because of potential applications in soil physics, geo-hydrology, oil extraction, chemical engineering and biological systems. This study is based on numerical modeling of heat transfer by an average flow based on Darcy’s law, in a saturated porous medium by a Newtonian fluid. Many research work has been performed in this field, citing as examples the theoretical similar \cite{1, 2} and \cite{4, 5}, by highlighting the effects of some physical parameters and the variability of the boundary conditions of the domain studied.

In this work, we propose to study the influence of some physical parameters on convection around a heated vertical flat plate embedded in a saturated porous medium in the presence of an internal heat source.
2 Mathematical formulation of the problem

The problem is to study the convection around a heated vertical flat plate embedded in a saturated porous medium, in the presence of an internal heat source. The distribution of the plate temperature varies according to the relationship \( T_w(x) = T_\infty + A x^\lambda \), where \( T_\infty \) is the temperature away from the plate assumed constant, \( A \) is a positive constant, \( \lambda \) is the exponent of the temperature supposed constant. Cartesian coordinates \( x \) and \( y \) are measured respectively along and perpendicular to the plate.

Taking into account certain assumptions, the system of equations governing the flow hydrodynamics and heat transfer is:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
u \frac{\partial T}{\partial x} &= \frac{a \cdot \partial^2 T}{\partial y^2} + \frac{\varphi}{\rho C_p}
\end{align*}
\]

The boundary conditions associated with the problem are:

\[
\begin{align*}
y &= 0 \quad x \geq 0 \quad u = 0 \quad v = V_w \quad T = T_w(x) \\
y &= \infty \quad x \geq 0 \quad u = 0 \quad T = T_\infty
\end{align*}
\]

where \( u \) and \( v \) are respectively the velocity components along axes \( x \) and \( y \), \( T \) is the temperature of fluid and \( \varphi \) is the internal source of heat. The constants \( \nu, K, a, g \) and \( \rho \) are respectively, kinematic viscosity, permeability, thermal diffusivity, gravitational acceleration and density. \( C_p \) and \( \beta \) are respectively the specific heat at constant pressure and coefficient of thermal expansion, \( V_w = B x^{-\frac{1}{2}} \) is the lateral mass flux, where \( B \) is a constant. Applying the similarity transformation, such as:

\[
\psi(x, y) = \frac{\alpha}{\pi} f(\eta) \quad \eta = \frac{x}{a} \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}
\]

\[
Ra_a = \frac{\alpha \beta (T_w - T_\infty) x}{a \nu} \quad \varphi = \rho C_p \frac{a (T_w - T_\infty) Ra_a^{1/2}}{x^2} e^{-\eta}
\]

where \( Ra_a \) is the Rayleigh number, \( \psi \) is the stream function defined by: \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \), \( f \) and \( \theta \) are the dimensionless similarity functions.

After substitution and development, the system of equations (1), (2) and (3) can be written:

\[
\begin{align*}
f''(\eta) - \lambda f'^2(\eta) + 2 f(\eta) f''(\eta) + e^{-\eta} &= 0 \\
\eta &= 0 \quad f(0) = f_w \quad f'(0) = 1 \\
\eta &\rightarrow \infty \quad f'(\infty) = 0
\end{align*}
\]

where \( f_w = -\frac{2B}{\lambda + 1} (\frac{\nu}{\alpha a} \beta K A)^{1/2} \) is the suction/injection parameter.

The local heat flux on the surface of heated vertical plate is given by:

\[
Q_w = -K_t \left( \frac{\partial T}{\partial y} \right)_{y=0} = -K_t A^{3/2} \left( \frac{\alpha \beta K}{\alpha \nu} \right)^{1/2} x^{\frac{3\lambda - 1}{2}} \theta'(0)
\]

where \( \theta'(0) \) is the gradient of the temperature at the surface of the plate and \( K_t \) is the thermal conductivity.

The local Nusselt number is defined by:

\[
Nu_x = -\theta'(0) Ra_a^{1/2}
\]
The local friction coefficient is defined by:

\[ C_f = \frac{\tau_w}{\frac{1}{2} \rho u_{\infty}^2} = \frac{2 Ra_x^{3/2} f''(0)}{Pr Re_x^2} \]  \hspace{1cm} (10)

where \( Re_x \) is the local Reynolds number, \( Pr \) is the Prandtl number, and \( \tau_w \) is shearing stress and it defined by:

\[ \tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} = \mu \left( \frac{a}{x^2} Re_x^{3/2} f''(0) \right) \]  \hspace{1cm} (11)

The thickness of the boundary layer is given by the equation:

\[ \frac{\delta}{x} \approx \frac{\eta_T}{Ra_x^{1/2}} \]  \hspace{1cm} (12)

where \( \eta_T \) is the value of \( \eta \) at the edge of the boundary layer, conventionally defined as the place where \( \theta \) is equal to the value \( 10^{-2} \).

3 Numerical results and discussions

The nonlinear equation (6) subject to the boundary conditions (7) is solved numerically by using the fifth order Runge-Kutta scheme associated with an iterative method. Solutions are obtained for distinct values of the temperature exponent parameter \( \lambda \).

We note that the suction corresponding to \( f_w > 0 \), injection to \( f_w < 0 \), and \( f_w = 0 \) to impermeable plate. The sign of the temperature gradient \( \theta'(0) \) determines the direction of the heat flow. \( \theta'(0) < 0 \) corresponds to the heat flows from the wall to the porous medium, \( \theta'(0) > 0 \) corresponds to the heat flows from the porous medium to the wall. The situation when \( \theta'(0) = 0 \) corresponds to an adiabatic surface where no heat transfer takes place.

Figure (a) shows the velocity \( f'(\eta) \) or temperature \( \theta(\eta) \) profiles across the boundary layers for \( \lambda = -\frac{1}{3} \) and for different values of the suction/injection parameter \( f_w \) with internal heat generation. It appears clearly that the surface of the plate becomes adiabatic for \( (f_w)_c = 2.99950 \). For any \( f_w > (f_w)_c \) (suction) the surface heat flow is always positive and it is directed from the plate to the porous medium. On the other hand, for \( f_w < (f_w)_c \) (injection) the temperature profiles have maxima where the heat flow is transferred from the porous medium to the plate. As \( f_w \) decreases from \( (f_w)_c \) to \( -\infty \), the height of the maxima tends to \( \infty \).

Figure (b) presents the velocity \( f'(\eta) \) or temperature \( \theta(\eta) \) distributions across the boundary layers for an isothermal plate \( \lambda = 0 \) and for different values of the suction/injection parameter \( f_w \) with internal heat generation.

We observe, in this case that the critical value where the surface is almost adiabatic is occurred at \( (f_w)_c = 0.86749 \). For \( f_w > (f_w)_c \), heat is transferred from the wall to the porous medium and for \( f_w < (f_w)_c \), heat is reversed and transferred to the wall from the porous medium.

Figure (c) shows the distributions of velocity or temperature in the case of a uniform heat flux to the plate \( \lambda = \frac{1}{3} \) and for various values of the suction/injection parameter \( f_w \) with internal heat generation. In this figure the surface heat transfer is directed from the surface to the convecting fluid for \( f_w > (f_w)_c = -0.54450 \), where at this critical value the surface is adiabatic. However for \( f_w < (f_w)_c \), heat is directed towards the surface from the porous medium.

Figure (d) illustrates the profiles of velocity or temperature in the case of uniform lateral mass flux to the plate \( (V_w = B) \) and a temperature that varies linearly with \( x (\lambda = 1) \) and for different values of the suction/injection parameter \( f_w \) with internal heat generation. The heat transfer occurs only in the forward direction for \( f_w \geq -1 \).

In the figures (e) and (f), it is clear that the thickness of the boundary layer decreases with suction and increases with the injection.
Figure (g) presents the effect of suction/injection parameter on the average friction coefficient along the plate and on the average Nusselt number for an isothermal plate. We can see that the heat transfer increases with suction but the average friction coefficient decreases.

Figure (h) shows the variation of the average friction coefficient along the plate and the average Nusselt number with some values of the temperature exponent $\lambda$ for an impermeable plate. One can see that the heat transfer increases with the temperature exponent $\lambda$ and the average friction coefficient along the plate decreases.

The temperature gradient at the surface of the plate has been compared to earlier works. The comparative statements have been presented in table 1 and 2. Although they are not significant variations in the compared values. However, it could be said here with a positive note that the method used in this current model is much simpler, faster and more accurate when compared to the method adopted by previous work.

(a) Profile of velocity or temperature for $\lambda = -\frac{1}{3}$ (b) Profile of velocity or temperature for $\lambda = 0$ and for different values of $f_w$ with internal heat source.

(c) Profile of velocity or temperature for $\lambda = 1/3$ (d) Profile of velocity or temperature for $\lambda = 1$ and for different values of $f_w$ with internal heat source.
(e) Profile of the thickness of the boundary layer for $\lambda = 0$ and for different values of $f_w$ with internal $\lambda = 1/3$ and for different values of $f_w$ with internal heat source.

(g) Effect of suction/injection parameter on the average friction coefficient along the plate and on the average Nusselt number for an isothermal plate.

(h) Effect of the power of the temperature on the average friction coefficient along the plate and on the average Nusselt number for an impermeable plate.

Table 1: Comparison of the temperature gradient at the surface of the plate with previously published results for impermeable heated vertical plate embedded in a saturated porous medium in the presence of an internal heat source.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$f''(0) = \theta'(0) = -Nu_{\infty} Ra^{-1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Postelnicu &amp; Pop</td>
<td>Mohamed E. Ali</td>
</tr>
<tr>
<td>Keller-box</td>
<td>RK 4</td>
</tr>
<tr>
<td>$-\frac{1}{3}$</td>
<td>0.99961</td>
</tr>
<tr>
<td>$-\frac{1}{4}$</td>
<td>0.67917</td>
</tr>
<tr>
<td>0</td>
<td>0.21524</td>
</tr>
<tr>
<td>$\frac{1}{7}$</td>
<td>-0.11415</td>
</tr>
<tr>
<td>1</td>
<td>-0.52409</td>
</tr>
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</table>
Table 2: Comparison of the temperature gradient at the surface of the plate with previously published results for permeable heated vertical plate embedded in a saturated porous medium in the presence of an internal heat source.

<table>
<thead>
<tr>
<th>$f_w$</th>
<th>$f''(0) = \theta'(0) = -N u_x Ra_x^{-1/2}$</th>
<th>$\lambda = 0$</th>
<th>$\lambda = -\frac{1}{3}$</th>
<th>$\lambda = 1$</th>
</tr>
</thead>
<tbody>
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<td></td>
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<td>Postelnicu &amp; all</td>
<td>P. Resultats</td>
<td>Postelnicu &amp; all</td>
</tr>
<tr>
<td>-1.0</td>
<td>RK 5</td>
<td>Keller-box</td>
<td>RK 5</td>
<td>Keller-box</td>
</tr>
<tr>
<td>0.36541</td>
<td>0.3658</td>
<td>0.06628</td>
<td>0.0662</td>
<td>-0.25508</td>
</tr>
<tr>
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<td>0.3184</td>
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<td>0.2882</td>
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<tr>
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<td>0.1253</td>
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<tr>
<td>0.6</td>
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<td>0.0744</td>
<td>-0.28697</td>
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</tr>
<tr>
<td>1.0</td>
<td>-0.03916</td>
<td>-0.0387</td>
<td>-0.42891</td>
<td>-0.4288</td>
</tr>
</tbody>
</table>

4 Conclusion

The numerical similarity solutions are obtained for equations (6) - (7). Special cases are considered for the temperature exponent $\lambda$ with the lateral mass flux controlled by the suction/injection parameter $f_w$. The case $\lambda = -1/3$, shows that at high injection ($f_w$ tends to $-\infty$), the temperature profiles show maxima where the heat is transferred from the porous medium to the plate. $\lambda = 0$ corresponds to a uniform surface temperature (isothermal plate $T_w = cte$) where the heat is transferred from the porous medium to the plate for $f_w < (f_w)_c$ ($f_w)_c = 0.8$ adiabatic plate) and in the opposite direction for $f_w > (f_w)_c$. The case $\lambda = 1/3$ corresponds to a uniform heat flux ($Q_w = cte$) at the surface of the plate. Here the surface of the plate is adiabatic for $(f_w)_c = -0.54$ ($\theta'(0) = 0$). $\lambda = 1$ corresponds to a uniform lateral mass flux to the plate where the heat is transferred only in the forward direction. Finally, it appears clearly that the boundary layer thickness decreases by suction and increases by injection.

Références