Creating non-planar static interfaces with magnetic fields

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 Résumé :
Notre équipe a récemment initié l'utilisation de la lévitation magnétique pour l'expérimentation sur les instabilités d'interfaces sous accélération (e.g. Rayleigh-Taylor). Dans ce contexte, nous avons développé une méthode permettant, à l'aide d’un champ magnétique modulé spatialement, de déformer de manière statique l’interface entre deux fluides selon une forme quasi arbitraire. Nous présenterons ici nos premiers résultats théoriques et leur application à notre installation expérimentale.

Abstract :
Our team pioneered the use of magnetic levitation to experiment on gravity-driven fluid interfaces instabilities (e.g. Rayleigh-Taylor). In this context, we developped a method which allows us to create non-planar static interfaces of arbitrary shapes between two fluids, by relying on spatially modulated magnetic fields. In this article, we introduce our first theorical results and their applications to our experimental technique.

Keywords : Interface Instability, Magnetic Levitation

1. Introduction

Interface dynamics is a wide field of research in fluid mechanics, both theoretically and experimentally. Although numerical simulations allow the exploration of more or less arbitrary configurations, experiments in the field have classically proven difficult to conduct, for many reasons linked to the physics of interfaces (complex surface tension phenomena, hydrodynamic instabilities, …). One of the most simple yet most pervasive phenomena preventing the experimental exploration of many configurations is gravity: At rest, two fluids separated by an interface must be in hydrostatic equilibrium. This limits the accessible experimental configurations to those in which the initial condition for the interface is flat, horizontal and stable. In order to circumvent this limitation, a member of our group pioneered the use of magnetic levitation of classical fluids [1]. By doping water with MnCl₂, the susceptibility $\chi$ of a fluid can be changed from diamagnetic to paramagnetic and its magnitude increased drastically. In the presence of a magnetic field $H$, the magnetic energy per unit volume becomes $-\frac{1}{2} \chi \mu_0 H^2$ (where $\mu_0$ is the permeability of free space), leading to a magnetic body force equal to the gradient of that energy. This force acts as a local “virtual gravity” added to Earth’s natural gravity. In the first attempts at using this approach in order to experiment on fluid interfaces, the parametric stability of liquid bridges was explored. A magnetic field $H$ was created such that Grad($H^2$) was as homogeneous as possible, thus leading to a net magnetic body force added to gravity. With a sufficiently large magnetic field gradient, it was then possible to compensate Earth’s gravity completely, creating an effective situation of microgravity which allowed the stabilization of perfectly cylindrical liquid bridges [1]. Changing the field intensity over time was then equivalent to applying carefully-controlled g-jitters [2]. More recently, that same idea has been applied to the problem of Rayleigh-Taylor instability (RT), where two fluids of different densities separated by an initially flat interface are subjected to gravity. Using the technique of magnetic levitation, our group was able to stabilize an initially RT-unstable interface using a magnetic force, and then abruptly “plunge” the system in an unstable configuration by rapidly turning off the current in the magnet. RT experiments were thus conducted with an almost perfect control over the initial conditions [3]. In the above examples, the magnetic field was designed so as to produce a spatially homogenous force whose intensity can be varied over time. Another course of action is possible, in which a spatial modulation of the force is applied. Two fluids of different densities and/or magnetic susceptibilities
subjected to gravity and the magnetic force will reach a purely hydrostatic equilibrium. If the combination of these forces is variable over space, then the system reaches an equilibrium in which the static interface is no longer planar (this equilibrium always exists since both the magnetic and gravity forces derive from a potential). This observation suggests that it is possible to create almost arbitrarily-shaped static interfaces between fluids of different densities and/or magnetic susceptibilities, and use these non-planar interfaces to experiment on interface physics or dynamics. This idea is explored in the present work, first by citing a successful proof-of-concept based on the RT instability, and then by estimating theoretically the range of static interface deformations that could be achieved with an equivalent set-up.

2. Drawing non-planar static interfaces

2.1 Proof-of-concept: the RT instability

As cited in the introduction, our group succeeded in conducting RT experiments with a very high control over the initial flatness of the unstable interface by relying on a stabilizing homogenous magnetic force that was turned off abruptly [3]. The magnetic field was produced by a coil magnet, to which were added two specially-designed Faraday pole pieces. The role of the pole pieces was to create field gradients such that there exists a spatial domain where \( \text{Grad}(H^2) \) is almost constant, leading to a homogenous net body force. Beyond that first attempt, we designed a system in which the magnetic field is periodically disturbed near the fluid interface, leading to a spatially-periodic body force [4]. This was done by placing magnetically permeable wires on both sides of the fluid cell, whose role was to act as “attractors” for the main magnetic field created by the pole pieces. The set-up is shown in Figure 1. Thanks to this design, it was possible to create a periodically-deformed static interface between two RT-unstable fluids (“RT-unstable” referring to the situation when only Earth’s gravity is present). Once these non-planar initial conditions were created, the RT instability could be produced simply by turning off the magnet. On a very short time (around 65 ms), both the large homogeneous body force and the periodic component disappeared, leaving the system in a configuration of pure RT instability with a periodically deformed interface as an initial condition.

Photographs excerpted from the videos are presented in figure 2. In these experiments, the Atwood number between the fluids was 0.29. The first line of photographs shows the destabilization of the interface without wires, where the average wave-length of the most unstable perturbation is 0.69 ± 0.05 cm (compared with 0.71 cm as predicted by the linear stability theory). The second line of photographs shows the same fluids in the same configuration, with wires perturbing the initial magnetic field. The wires were sinusoidally-shaped with a period of 2.25 cm and an amplitude of 0.3 cm. Although the initial deformation is too small to be visible on the images, a regular periodicity can be observed in the falling spikes on the images with wires. This first attempt demonstrated the validity of the concept, and the relevance of using permeable wires as local attractors in order to perturb the average magnetic body force. This first experiment paved the way for a more systematic exploration of the effect of such wires on the shape of the initial interface. This analysis is presented in the next section.
2.2 Formalization of the theoretical problem

We present here the analysis of the influence of the permeable wires on the initial static interface shape. A magnetic fluid of density $\rho = \rho_f$ and of susceptibility $\chi = \chi_f$ is maintained by a magnetic field $H(y, z)$ above a lighter fluid of density $\rho = \rho_l < \rho_f$ and susceptibility $\chi = \chi_l < \chi_f$. The interface profile $z = \eta(y)$, separating fluid 2 ($z < \eta(y)$) from fluid 1 ($z > \eta(y)$) in the domain $[L_1, L_2]$, is such that the local curvature is related to the jump in pressure between the two fluids (Laplace law):

$$\sigma \kappa = p_1 - p_2$$

where $\sigma$ is the interfacial tension between the two fluids, $\kappa$ the local curvature defined by $\kappa(\eta(y), \eta'(y)) = \eta'(y)/[1 + (\eta'(y))^2]^{1/2}$ and $p_k$ ($k \in \{1, 2\}$) the pressure in fluid k. Relating $p_k$ to the magnetic and gravity volumetric potential energies yields:

$$\sigma \kappa + (\rho_l - \rho_2) g \eta + \frac{1}{2} \mu_0 (\chi_2 - \chi_1) H^2 = 0$$

The system is made dimensionless by the following choice of variables:

$$\bar{y} = y/L_2, \quad \bar{z} = z/L_2, \quad \bar{\eta} = \eta/L_2, \quad \bar{p}_k = \rho_k/\rho_{ref}, \quad \bar{H}^2 = \frac{H^2}{(H^2)_{ref}} \quad \text{where} \quad \rho_{ref} = (\rho_1 + \rho_2)/2$$

and by introducing the three following dimensionless numbers: $C$, a number expressing the contrast of length, $B_0$ the Bond number and $\alpha$, a number expressing the ratio of magnetic forces to gravitational forces:

$$C = L_2/L_1, \quad B_0 = \rho_{ref} g L_2^3/\sigma, \quad \alpha = \mu_0 (H^2)_{ref}/(\rho_{ref} g L_2).$$

With these definitions the dimensionless equation becomes:

$$C^2 \kappa + B_0 \left[ (\bar{p}_1 - \bar{p}_2) \bar{\eta} + \frac{1}{2} \alpha (\chi_2 - \chi_1) \bar{H}^2 \right] = 0$$

In addition, the average height of the interface is determined by the value of the total volume $V$ of the fluids in the cell. This yields the condition: $\int_0^L \bar{\eta}(\bar{y})d\bar{y} = \bar{\eta}_{ref}$, where $\bar{\eta}_{ref}$ is a fixed quantity function of $V$.

2.3 Resolution using the Relaxation method

Here we first solve equation (3) using a relaxation method. In discretized form, the $n$ equations to be solved $\bar{\eta}_i$ are for each $\bar{\eta}_i$ defined at regularly spaced values of $\bar{y}_i = \bar{y} \times i, i \in \{0..n - 1\}$, with the boundary condition such that $\bar{z} = \bar{\eta}(\bar{y})$ is periodic (i.e. $\bar{\eta}_{n-1} = \bar{\eta}_0$) and with the constraint imposed by the mass conservation (yielding the value of $\bar{\eta}_{ref}$):

$$C^2 \kappa(\bar{\eta}_i, \bar{\eta}_n, \bar{\eta}_{i+1}) + B_0 \left[ (\bar{p}_1 - \bar{p}_2) \bar{\eta}_i + \frac{1}{2} \alpha (\chi_2 - \chi_1) \bar{H}^2(\bar{y}_i, \bar{\eta}_i) \right] = \epsilon_i \quad \bar{\eta}_0 = \bar{\eta}_{n-1} \quad \frac{1}{n} \sum_{i=0}^{n-1} \bar{\eta}_i = \bar{\eta}_{ref}$$. 

FIG. 2 - Images of interface deformation (where $t = 0$ is the time at which the magnetic force is turned off). First line : without wires ; second line : with wires.
The relaxation method consists in reformulating the equations to be solved as coupled diffusion equations, $d\bar{\eta}_i/dt = \varepsilon_i$, and perform iterations from an initial trial configuration until convergence. The choice of the discretization operator for the time derivative characterizes the particular relaxation scheme. The formulation used here is the Crank-Nicholson scheme defined by:

$$\bar{\eta}_i^{p+1} = \bar{\eta}_i^p + \Delta t \left( \varepsilon_i^{p+1} + \varepsilon_i^p \right) / 2$$

(4)

where $p$ is the number of iteration. In order to impose the constraint, the interface equations must be supplemented by an auxiliary function with an associated Lagrange multiplier: $\varepsilon_i' = \varepsilon_i + \lambda \partial V / \partial \bar{\eta}_i$.

Since $V$ is linear in $\bar{\eta}_i$, this is equivalent to the addition of a constant to equation (4). The constant is found after solving (4) by taking the difference between the proposed new average height and the imposed average height: $\lambda = (\bar{\eta}_{\text{ref}})_{\text{computed}} - (\bar{\eta}_{\text{ref}})_{\text{imposed}}$.

### 2.4 The approximation for $\overline{H^2}$

The exact value of $H^2$ everywhere in the experimental domain is an involved numerical computation, especially with a three-dimensional wire inserted in the design. It can be conducted by solving magnetostatic equations, for instance using finite elements. Given the limited width of the fluid cell (along the $x$ direction as defined in figure 1), the only force of interest is that produced by the magnetic field in the symmetry plane of the device (namely, the symmetry plane of the cell). In order to avoid having to calculate a full 3D magnetostatic problem for each wire geometry, we investigated the possibility of using approximate expressions of $\overline{H^2}$ in the mid-plane of the fluid cell. We relied on the finite-element package Cast3m, developed by CEA (France). By doing systematic 2D and 3D calculations of the magnetic field with and without wires, we showed that the non-dimensional quantity $\overline{H^2}$ can be approximated as the sum of two contributions: $\overline{H^2}_{\varphi}$, the contribution due to the Faraday pole pieces in the absence of wires, and $\overline{H^2}_z$, the correction due to the wires. $\overline{H^2}_{\varphi}$ is designed so as to produce an homogeneous vertical force along the $z$ axis, so to first order, $\overline{H^2}_{\varphi}$ is linear in $z$. A very good fit with full-3D calculational results can be obtained by assuming that $\overline{H^2}_z$ behaves like a Gaussian whose argument is a function of the vertical distance to the projection of the wires in the mid-plane of the fluid cell. For wires of equations $z = \bar{z}_{\text{wire}}(\bar{y})$, we can write:

$$\overline{H^2}_z(\bar{y},\bar{z}) = c \exp\left[-\gamma\left(\bar{z} - \bar{z}_{\text{wire}}(\bar{y})\right)^2\right] \quad \text{and} \quad \overline{H^2}(\bar{y},\bar{z}) = a\bar{z} + b + c \exp\left[-\gamma\left(\bar{z} - \bar{z}_{\text{wire}}(\bar{y})\right)^2\right]$$

For a sinusoidally-shaped wire, one has of course: $\bar{z}_{\text{wire}} = \bar{z}_0 + A \sin(2\pi \bar{y})$.

The above approximation for $\overline{H^2}$ has been found to be valid within 3% of the complete numerical calculation for rectilinear wires, and for a wide range of different wire diameters. This approximation naturally extends to wires deformed in such a way that their slope remains small. We are presently conducting a more systematic series of 2D and 3D calculations in order to explore excursions away from this geometrical condition. Figure 3 shows the result of a 2D calculation conducted for rectilinear wires of radius of 0.44 mm and of a relative permeability of 2.6, in the same geometry as the one used for the RT experiments above.

FIG. 3 - domain of influence of the wires on the squared magnetic field; black dots are for locations where $\overline{H^2}_z/\overline{H^2}_\varphi > 1\%$, gray dots for the others.
The domain represented in the figure is a portion of the computational domain (the lower right-hand quarter). \( x = 0 \) represents the mid-plane of the fluid cell (the plane of interest for the force calculation), while the tilted boundary on the right-hand side is the right Faraday pole piece. In the figure, colors refer to the ratio \( H^2_w / H^2_p \) over \( H^2_p \). Black points represent locations where \( H^2_w / H^2_p > 1\% \), gray points represent locations where that ratio is lower than 1\%. The black zone shows the spatial domain, which is effectively perturbed by the presence of the wires. Note the 4-lobe shape and the fact that the left-hand lobe crosses the mid-plane of the cell (as it should).

2.5 Results

A first application of the above procedure has been made by selecting a wire shape — here, a sinusoid of fixed amplitude — and calculating the associated interface shape for different values of the average height \( \overline{H}_{\text{ref}} \) (which physically correspond to placing the fluid cell higher or lower between the wires and the pole pieces). As a result of these different positions of the cell, different interface profiles can be formed. As an example, we have examined the conditions of the experiment presented in Huang et al. [4]. Fluid 1 is a heavier paramagnetic aqueous mixture and fluid 2 is hexadecane (see reference for physical parameters). The corresponding dimensionless parameters are \( \overline{\rho}_1 = 1.286 \), \( \overline{\rho}_2 = 0.7134 \), \( \chi_1 = 7.29 \times 10^{-4} \), \( \chi_2 = 0 \), \( Bo = 459.4 \), \( C = 0.6533 \), \( H^2_{\text{ref}} = \rho_{\text{ref}} g L_z / \mu_0 = 1.243 \times 10^{-8} \) and \( \alpha = 1 \). The radius of the wires is 0.44 mm and its sinusoidal shape is defined by an amplitude of 1 mm, a period of 2.25 cm and an average level \( z_0 = 0.5132 \).

Figure 4 shows the results for ten different values of \( \overline{H}_{\text{ref}} \), ranging between 0 and 1 (the experimental case corresponds to \( \overline{H}_{\text{ref}} = z_0 = 0.5132 \)). If now we use a more complex wire shape (for instance adding a harmonic as follows: \( z_{\text{wire}} = z_0 + A \sin(2\pi \gamma) + B \sin(4\pi \gamma) \)), then more complex interface shapes can be created, like shown in figure 5. Besides changing the wire shape, another control parameter is the relative permeability. A wire of higher relative permeability, just as a wire of larger radius, will yield stronger force perturbations.

**FIG. 4** - several interface profiles obtained for the same sinusoidal wire (white dots) and for different vertical positions of the cell in the mid-plane of the device (\( \overline{H}_{\text{ref}} \) from 0.1 to 0.9 by steps of 0.1, from bottom to top).

**FIG. 5** - interface profile for a wire shape based on two sine harmonics.
2.6 The energetic approach

Another way to determine the static interface profile between the two fluids involves minimizing the total energy of the system. The energy of the two fluids subjected to the magnetic force is the sum of two contributions: the potential energy (gravitational and magnetic) and the interfacial energy. After some algebra, one can establish that the dimensionless total energy can be expressed as follows:

\[
\bar{E}(\bar{\eta}) - \bar{E}(\bar{\eta}_{ref}) = \int_0^1 F(\bar{\eta}) d\bar{y}
\]

\[
F = B_0 \left[ (\nabla \cdot \bar{\eta} - 1)^2 + \frac{1}{2} \alpha (\chi - \chi_0)^2 + \frac{1}{2} \alpha (\chi - \chi_0)^2 \right] + \left[ 1 + (C \bar{\eta})^2 \right]^{-1}
\]

with the two following constraints:

\[
\int_0^1 \bar{\eta}(\bar{y}) d\bar{y} = \bar{\eta}_{ref}, \quad \bar{\eta}(0) = \bar{\eta}(1)
\]

The basic problem is thus to minimize the quantity \( \bar{E}(\bar{\eta}) - \bar{E}(\bar{\eta}_{ref}) \), taking into account the two constraints in order to find the interface profile. We presently are working on a method relying on the Fourier decomposition of the interface optimized with the Nelder-Mead Simplex method [5]. One potential benefit of this alternate approach, besides being a validation of the relaxation method, will be to help develop an improved formulation of the inverse problem (see section 3). Our first results show that the same interface shapes are predicted as with the relaxation code, thus leading to a cross-validation of both approaches.

3. Conclusions and perspectives: the inverse problem

We demonstrated a novel experimental technique enabling us to create static non-planar interfaces of potentially complex shapes. Through a systematic numerical analysis, we explored quantitatively the range of accessible interface deformations through this technique. Beyond its first application to the problem of controlling the initial deformation of a RT-unstable interface, this experimental technique could also be used in a much wider range of experimental explorations, where the ability to force an arbitrary shape to a static interface would be a powerful experimental tool. We validated the technique itself in a qualitative way through the analysis of the RT instability. The quantitative validation of the numerical exploration is currently being made, its main objective being a direct comparison of the experimentally observed interface shapes with the calculated profiles. Beyond this initial validation, a highly promising development of our technique lies in the resolution of the corresponding inverse problem. In the above numerical study, we determined the shape of an interface based on the predetermined shape of permeable wires affixed on each side of the fluid cell. Our objective now is to solve the inverse problem: Given an arbitrarily chosen interface shape, what kind of permeable wires should be placed in the experiment in order to create the corresponding interface? We are presently exploring theoretical procedures to solve that new problem. Our aim is to provide experimenters with an easy-to-use and fast numerical tool, allowing the experimenter to draw an arbitrary non-planar interface, and then yielding the corresponding wire shape to use in the experiment.

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