SUMMARY

The eigenfrequencies of a cylindrical obstacle (of finite or infinite length) can be interpreted as the resonances due to phase matching of circumnavigating helical surface waves. For the case of a cylinder of finite length, the pitch angle of the helix can assume a discrete set of values only. Resonant eigenvibrations can be excited by waves incident in an oblique fashion, which generates the helical waves. A refraction effect is found to take place between the incident and the helical-wave directions. We obtain pole diagrams of the scattering amplitude in the complex-frequency plane, by using the T-matrix approximation for finite cylinders. In addition, pole diagrams for spheroidal scatterers are obtained by the use of the T-matrix and of spheroidal wave functions. While the poles of symmetric scatterers (spheres or infinite cylinders) are degenerate in the azimuthal quantum number $m$, the degeneracy for the poles of finite cylinders and of spheroids is lifted. This $m$-splitting is explained by the phase matching of helical waves with various allowed pitch angles. Dispersion curves for the phase and group velocities and attenuations of the helical waves are obtained.

KEY WORDS

Infinite cylinders, finite cylinders, spheroids, eigenfrequencies, resonances, helical surface waves, oblique incidence, complex-frequency poles, $m$-splitting, dispersion curves, phase velocities, group velocities.
RÉSUMÉ

Les fréquences propres d'un obstacle cylindrique (de longueur finie ou infinie) peuvent être interprétées comme dues à l'accord entre les phases d'ondes se propageant sur la surface d'une façon hélicoidale. Dans le cas d'un cylindre de longueur finie, l'angle de pas de l'hélice ne peut prendre qu'une série de valeurs discrètes. Des vibrations propres résonnantes peuvent être excitées par des ondes incidentes de direction oblique, ce qui produit les ondes hélicoidales. Un effet de réfraction est trouvé entre les directions de l'onde incidente et de l'onde hélicoidale. On obtient des diagrammes de pôles de l'amplitude de diffusion dans le plan complexe de la fréquence, par un calcul utilisant l'approximation de la matrice T pour des cylindres finis. En plus, on obtient des diagrammes de pôles pour des obstacles sphéroïdaux en utilisant la matrice T, ou des fonctions d'ondes sphéroïdales. Tandis que les pôles d'obstacles symétriques ( sphères, ou cylindres infinis) dégénèrent vis-à-vis du nombre quantique azimuthal m, cela n'est plus le cas pour les pôles de cylindres finis et de sphéroïdes. La séparation résultante entre les valeurs de m s'explique alors par l'accord de phases des ondes hélicoidales possédant différents angles d'inclinaison permis. On obtient des courbes de dispersion pour les vitesses de phase et de groupe des ondes hélicoidales.

MOTS CLÉS

Cylindres finis, cylindres infinis, sphéroïdes, fréquences propres, résonances, ondes de surface hélicoidales, incidence oblique, pôles de fréquence complexe, écartement (dédoublement) en m, courbes de dispersion, vitesse de phase, vitesse de groupe.

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1. Introduction

The most general case of the scattering of waves obliquely incident on an infinite cylinder has first been considered by White [1]; he deals with a plane wave in an elastic medium incident on an elastic cylindrical inclusion. This theory has recently been completed by Delsanto et al. [2, 3]; it is based on the classical normal-mode (or partial-wave) Rayleigh series expansion. Special cases have been investigated in greater detail: acoustic waves incident on an elastic cylinder [4] or shell [5], or on a fluid cylinder [6, 7], and elastic waves incident on a fluid-filled cylindrical cavity [8] (in a similar fashion, elastic waves on a spherical cavity have also been studied [9]). In these latter investigations, the complex eigenfrequencies of the mentioned obstacles have been explicitly calculated, as well as the excitation of the corresponding eigenvibrations. While analogous calculations cannot be carried out for the case of cylinders of finite length, due to the non-separability of this problem, results for the eigenfrequencies can nevertheless be obtained here by the use of special methods. If only the interior vibrations of a fluid cylinder in vacuo are considered, the problem is still exactly soluble, and the corresponding eigenvibrations have, for the first time, been interpreted as being due to the phase matching of helical surface waves [10] (of internal type, in this case). For the non-separable exterior problem of finite-length impenetrable cylinders, the T-matrix method of Waterman [11] has been employed, modified so as to furnish complex eigenfrequencies [12]. A very comprehensive study of this problem has recently been carried out, on the basis of this method, which is contained in a paper [13] that also discusses the acoustic eigenfrequencies of impenetrable spheroids obtained by the use of spheroidal functions. In this study, the phase matching of external helical surface waves has been invoked as an explanation for the finite-cylinder eigenfrequencies, and for their splitting into components corresponding to different values of the azimuthal quantum number m (the latter being a measure for the pitch angles of the helical waves, which due to the finite cylinder length form a discrete set). As to the excitation of helical surface waves by incident sound, it was found [6, 13] that this takes place in a refractive way, the helical pitch angle being different from the incident angle. Experiments on the scattering of obliquely incident sound by elastic cylinders have recently been initiated [14].

2. Internal Helical Waves

A fluid cylinder in vacuo, of radius a and length L admits an internal acoustic field (wave vector k, with \( k = \omega/c \)):

\[
(2.1) \quad p(r) = J_m(K, r) e^{\pm ik_a z} + e^{\pm ik_z z},
\]

where:

\[
(2.2a) \quad K^2 = k^2 - k_z^2;
\]
the single-valuedness condition under $\varphi \rightarrow \varphi + 2\pi$
gives:

\[(2.2.b) \quad k_{q}=m/a, \quad m=1, 2, \ldots,\]

and the boundary condition at the end faces leads to:

\[(2.2.c) \quad k_{q}=[\pi/L, \quad j=1, 2, \ldots,\]

The Dirichlet boundary condition at $r=a$ gives:

\[(2.2.d) \quad K=x_{mn}/a, \quad n=1, 2, \ldots,\]

where $x_{mn}$ is the $n$-th zero of $J_{m}(x)$. [The Neumann
condition for a fluid cylinder in a rigid enclosure would lead to the zero of $Y_{m}(x)$] Inserted in equation
\[(2.2.a),\]
this gives the eigenvalues $k_{nmj}$ of $k$.

We now introduce a tangential wave vector $k_{t}$ of the
surface field, with:

\[(2.3) \quad k_{t}^{2}=k_{q}^{2}+k_{z}^{2},\]

which describes the propagation of helical surface
waves. The conditions \[(2.2.b)\] and \[(2.2.c)\] then represent
the phase matching of such waves after circumnavigating the cylinder, and/or getting reflected from the end faces. Equations \[(2.2.b),\] \[(2.2.c)\] inserted in equation \[(2.3)\] furnish eigenvalues \((k_{t})_{nmj}\).

The helical-wave phase velocities $c_{s}=\omega/k_{s}$ can be obtained from $c_{s}/c=1/k_{s}$ at the discrete points (resonance frequencies) where phase matching is satisfied:

\[(2.4) \quad c_{s}/c)_{nmj}=[(z_{nm}+(j \pi)(a/L)^{2})]/\]

\[(m^{2}+(j \pi)^{2}(a/L)^{2})]^{1/2}.\]

Note that these correspond to helical waves of pitch angle $\alpha$ with the $z$-axis, $\tan \alpha=k_{t}/k_{z}$, which due to the finite length of the cylinder assumes the discrete values:

\[(2.5) \quad \tan \alpha_{nmj}=(m/j \pi)(L/a).\]

A given helical wave thus corresponds to a fixed ratio $m/j$. For an infinite cylinder \((L \rightarrow \infty, j \rightarrow \infty)\) $\alpha$ is continuous.

Figure 2.1 shows how the discrete points of equation \[(2.4),\]
corresponding to the eigenfrequencies of the cylinder, when connected according to equation \[(2.5)\] furnish the dispersion curves of the helical surface waves.

3. Refraction effect

For a cylinder in a medium helical waves can be generated by an incident plane acoustic wave. If the latter arrives at an angle $\gamma$ with the $z$-axis so that $k_{z}=k \sin \gamma$, $k_{r}=k \cos \gamma$, the total field for an infinite cylinder, given by equation \[(2.2.c)\] of reference [15], gets modified to:

\[(3.1) \quad p_{v}^{n}=\frac{1}{2} e^{ik_{z}z} \sum_{x=0}^{\infty} (2-\delta_{x0}) P_{x} \left[H_{n}^{(2)}(k_{r}r) \right]

+\sin \varphi,\]

leading after application of the Watson transformation
to the creeping-wave sum analogous to equation \[(2.10c)\] of [15]:

\[(3.2) \quad p_{v}=e^{ik_{v}z} \sum_{x=1}^{\infty} (2-\delta_{x0}) P_{x} \left[H_{n}^{(1)}(k_{r}r) \cos \varphi \right]

\times S_{n}(v_{i})^{2}+\varphi \Re v_{i},\]

The phase factor $\Phi_{i}=k_{z}z+\varphi \Re v_{i}$ shows that these surface waves are helical, with wave fronts $\varphi=-\left(ak_{r}/\Re v_{i}\right)z$ whose normals make an angle $\varphi_{i}=\tan^{-1}\left(\Re v_{i}/ak_{r}\right)$ with the $z$-axis. This defines the law of refraction:

\[(3.3) \quad \tan \varphi_{i}=g_{i} \tan \gamma, \quad g_{i}=\Re v_{i}/k_{r},\]

between incident direction $\gamma$ and helical-wave
direction $\varphi_{i}$. The phase velocity $v_{ph}^{i}=ck/k_{t}$ of the helical waves is:

\[(3.4) \quad v_{ph}^{i}=c/\left[(\Re v_{i}/ka)^{2}+\cos^{2}\gamma\right]^{1/2},\]

for the case of external waves on rigid or soft cylinders
where the asymptotic expansion of Franz [16] for $v_{i}(k_{r}, a)$ can be used, one has:

\[(3.5) \quad v_{ph}^{i} \approx c/\left[1+\frac{q_{i}}{2.6^{1/3}} \frac{\sin^{2}\gamma \left(ka\right)^{2/3}}{2/3}+\ldots\right],\]

where $q_{i}^{r}=1.469354$, $q_{i}^{r}=3.372134$. Including higher terms, figure 3.1 shows $v_{ph}^{i}$ of helical waves at
$\gamma=0^\circ$ and $45^\circ$, and the refraction angle $\varphi_{i}$ for $\gamma=45^\circ$, for a soft cylinder.

4. Complex eigenfrequencies

The eigenfrequencies $k_{nmj}$ for a cylinder in vacuo,
Section 3, are real since no radiation loss can occur.
We present figures showing examples of complex
5. Cavities

Analogous results were obtained for infinite cylindrical cavities, using our general theory [2, 3]. Figure 5.1 shows the eigenfrequencies in the complex $k, a$ plane for an empty cavity in aluminum, for both compressional ($p$) and shear ($s$) type surface waves (here, $k$ is the dilatational propagation constant).

6. Conclusion

The complex eigenfrequencies of finite-length cylinders show a splitting according to the azimuthal quantum number $m$. They can be interpreted as the resonances, due to phase matching, of helical surface waves of different pitch angles. For infinite cylinders, a continuum of pitch angles occurs. The helical surface waves can be excited by incident acoustic waves, and refraction takes place between the incident and the surface wave directions. Recent experiments [14] are now investigating these problems. Theoretically,
there have also been geometrical investigations of helical waves on cylinders [17]. Portions of this work were supported by the Office of Naval Research, the Army Research Office, the Naval Surface Weapons Center, and the David Taylor Ship Research and Development Center.

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