Diffusion

of an ultrasonic wave

by a simple rough surface

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SUMMARY

In order to study reflection and transmission of bounded ultrasound, most authors base their theory on the Fourier-transform method. The uniqueness of this solution is not proven neither the reason why the solutions are always taken in planes parallel to the reflecting plane, while measurements are always made in other planes. We developed a more exact study based on the black box-theory.

In a second part, periodically rough surfaces are treated by means of the plane-wave theory. Generation of surface waves by perpendicular incidence can be explained. Considering Bragg-diffraction we try to measure the peak-valley distance of the roughness.

KEY WORDS

Ultrasound, acousto, optics, diffractions, reflections, transmissions, nondestructive evaluation.

RÉSUMÉ

Presque toutes les théories concernant la transmission et la réflexion des ultrasons sont fondées sur la méthode de la transformée de Fourier. Lunicité de la solution n’est jamais prouvée, d’autre part on prend toujours la solution dans le plan de réflexion ou dans un plan parallèle, tandis que les mesures se font toujours dans des plans obliques. Un calcul plus exact est développé, basé sur la théorie de la boîte noire (black box).

Dans la seconde partie on a traité des surfaces rugueuses périodiques en utilisant des ondes planes et infinies. On explique l’existence d’ondes de surface en incidence normale. Pour l’angle de Bragg nous trouvons une relation entre la hauteur de la surface rugueuse triangulaire et l’intensité ultrasonore diffusée du premier ordre.

MOTS CLÉS

Ultrasounds, acousto, optics, diffractions, reflections, transmissions, evaluation non destructive.
1. Reflection and transmission of bounded ultrasonic beams

1.1. Some remarks about the fundamental theory

Most theoretical studies concerning reflection and transmission of bounded ultrasound, using Fourier-transforms, is based on the work of Brekhovskikh (1). Assuming that an incident beam, can be represented in the plane $z=0$ (fig. 1), by:

$$ F(x) = f(x)e^{ikx}, \quad f(x) = 0, \quad x \notin [-a, a] $$

and $f(x)$ being the amplitude distribution in the plane perpendicular to $k$ (wave vector of the incident beam, making an angle $\theta$ with $e_z$):

$$ k_i = (\omega/c)\sin \theta, \quad k \sin \theta = k_i, \quad k = \omega/c. $$

The partial differential equation describing the incident beam becomes:

$$ \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} + \frac{\omega^2}{c^2} \varphi = 0, $$

with boundary condition:

$$ \varphi(x, 0) = F(x). $$

Putting:

$$ V(k_x) = \int_{-\infty}^{\infty} F(x)e^{-ik_x x} dx $$

and:

$$ \varphi(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(k_x)e^{ik_x x + ik_z z} dk_x, $$

with:

$$ k_x^2 + k_z^2 = \omega^2/c^2. $$

We see that expression (6) describes the incident beam; because it is a solution of (3) and the boundary condition (4) is also fulfilled. In order to deduce the reflected and transmitted beam, we interpret this incident beam as being built up by means of an infinite number of plane waves, all having an amplitude $V(k_x, k_z)$. Thus after reflection each plane wave becomes:

$$ R(k_x) = V(k_x)e^{ik_x x - ik_z z}, $$

$R(k_x)$: plane-wave reflection-coefficient.

The total reflected beam is then given by:

$$ \varphi_{ref}(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R(k_x) V(k_x)e^{ik_x x - ik_z z} dk_x, $$

Also for the transmission we find:

$$ \varphi_{trans}(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} T(k_x) V(k_x)e^{ik_x x + ik_z z} dk_x, $$

$T(k_x)$: plane wave transmission-coefficient. $k_z$ being the $z$-component of the plane-wave in the new medium.

Remark:

(a) This reasoning is too simple, because other solutions can be built up, f. e.:

$$ \varphi(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} V(k_x)e^{ik_x x + ik_z z} dk_x + A \sin(kz). $$

(b) If differentiation and integration may be changed, as was done in above boundary problem, the solution should also yield in the planes $z = ax + b$, although most authors describe their solution in the planes parallel to $z = 0$! Why?
For this reasons we treat a more exact mathematical solution.

1.2. More exact treatment based on the black box theory

Starting from a given amplitude distribution \( f(x) \) in the plane \( z=0 \), we wish to calculate the distribution \( g(x) \) after a propagation over a distance \( d \). This problem being described by the Helmholtz equation:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0, \quad k = \frac{w}{v},
\]

with boundary condition:

\[ u(x, 0) = f(x) \]

and the Sommerfeld condition in infinity:

\[
-\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u}{\partial r} -iku \right) = 0,
\]

uniformly in \( \theta \) and for \( r \to \infty \).

The solution for this problem being known [2]:

\[
\varphi(x, z) = -\frac{1}{2} \int_{-\infty}^{+\infty} f(x') H_0^1(k, R) \, dx',
\]

and \( z > 0 \) and \( H_0^1 \) is the Hankel function.

We show that:
1. \( g(x) \) can be calculated from a convolution product:

\[
g(x) = w * f.
\]
2. \( w(x) \) can be calculated as a response of a harmonic input:

\[
f(x) = e^{ikx}
\]
deduced from a plane wave \( e^{ikx+ikz} \) in which \( z = 0 \).

3. The solution given by Brekhovski h only yields in planes \( z = d \).

Thus that the considered incident beam is only a representation.

From (15) we deduce that:

\[
g(x) = -\frac{1}{2} i (f * n)(x),
\]

with \( n(x) = (\partial/\partial z) H_0^1(k, \sqrt{x^2+z^2}) \), \( d > 0 \), \( z = d \), which states the convolution character 1.

From this convolution property we calculate the Fourier-transform, and find:

\[
(\mathcal{F}(g) = \mathcal{F}(w), \mathcal{F}(f) \text{ or } \mathcal{W}(k_x), \mathcal{V}(k_x)).
\]

Knowing \( W(k_x) \), an inverse transform gives us \( g(x) \).

In order to find \( W(k_x) \) we consider an input:

\[
f(x) = e^{ikx}
\]

then:

\[
g(x) = w * e^{ikx} = W(k_x) e^{ikx}.
\]

But for a plane wave \( g(x) \) is known, and given by \( g(x) = e^{ikx+ikz} \) from which we deduce:

\[
W(k_x) = e^{ikz}.
\]

This result is also easily found from:

\[
w(x) = -\frac{1}{2} n(x),
\]

in taking the Fourier transform, in which way we conclude that \( w(x) \) can be calculated from the response of a harmonic incident wave.

From (18) we find the wave form at a distance \( d \), is given by:

\[
(\mathcal{F}(g) = \mathcal{F}(w) \cdot \mathcal{V}(k_x), \mathcal{V}(k_x), \mathcal{F}(f).
\]

always in the supposition that the waveform is known at \( z = 0 \). The real problem remains; "How to transform the amplitude distribution in \( z = ax + b \) to the plane \( z = 0 \)." Once this problem is solved, then reflection and transmission can be treated by the blackbox theory described above, which gives the wellknown solutions for plane surfaces. The Fourier-transform-method however is also applied by several authors for periodically disturbed surfaces, although it is known that the convolution can only be found for translation invariant systems. And those translation invariance is only approximated by wide transducers. Again the solution is also only true in the planes parallel to the reflecting plane and never in \( z = ax + b \).

1.3. Bounded beam built up by means of a finite some of inhomogeneous beams

In order to avoid the Fourier-transform description we describe a bounded beam by means of a sum of inhomogeneous beams incident all in the same direction [3] and given in the form:

\[
\varphi(x, z) = \sum_{n=0}^{N} A_n e^{ikx} e^{-ikz} \, \mathbf{e}^2
\]

We calculated the projection of an incident Gauss beam as is done in the Fourier-transform method and compared this result with the numerical values deduced from (22) in a plane parallel with the reflecting plane. Both results were approximately the same as can be seen from Figure 2.
Calculating the reflection coefficient for one plane inhomogeneous beam, the total reflected beam will then be given by:

$$R(\beta_n)A_n e^{\beta_n x} e^{-k^2 x^2} e^{\beta_n z}.$$  

(23)

For more details concerning reflection and transmission we refer to [3].

The boundary conditions become now:

$$\nabla \cdot \mathbf{u} = \text{grad } h = u^z \cdot \text{grad } h,$$

on \( h(x, z) = 0 \) with \( h(x, z) = f(x) - z \).

(26)

$$T_L^H (\text{grad } h) j = T_R^H (\text{grad } h),$$

on \( h(x, z) = 0 \); 1: liquid; 2: solid.

2. Reflection of a plane wave on a periodic curved surface

Considering a solid surface described by \( z = f(x) = f(x + \lambda) \) as shown in Figure 3, and a plane wave incident making an angle \( \theta \) with the \( z \)-axis, having the wavelength \( \lambda \), the reflected and transmitted waves are governed by:

$$\frac{\partial^2 \varphi_m}{\partial x^2} + \frac{\partial^2 \varphi_m}{\partial z^2} + \frac{w^2}{\nu_m^2} \varphi_m = 0,$$

(24)

\( m = i \): incident wave; \( m = r \): reflected wave; \( m = d \): transmitted longitudinal wave; \( m = s \): transmitted transversal wave.

The proposed potential functions read:

$$\varphi_m(x, z) = \sum_{i} M_i e^{i(k_n x + \nu_m z)}.$$  

(31)

Following the reasoning used by different authors before [6], [7] we look for a solution described in a series of plane inhomogeneous waves.

We further transform above expressions taking into account [4] and [5] in:

$$\frac{\partial \varphi_r}{\partial z} - \frac{\partial \varphi_d}{\partial x} - f'(x) \left( \frac{\partial \varphi_2}{\partial x} - \frac{\partial \varphi_d}{\partial z} \right) = \left( k_x f'(x) - k_z \right) e^{i(k_n x + \nu_m z)},$$

(27)

$$\mu \left[ \frac{\partial^2 \varphi_r}{\partial x^2} + \frac{\partial^2 \varphi_d}{\partial z^2} - \frac{\partial^2 \varphi_j}{\partial x^2} \right] + f'(x) \left( k_x^2 \varphi_r - 2 \mu \left( \frac{\partial^2 \varphi_2}{\partial x^2} + \frac{\partial^2 \varphi_d}{\partial z^2} \right) \right) = \rho \left( k_x^2 f'(x) + \nu_m^2 \varphi_r \right),$$

(28)

$$\mu \left[ \frac{\partial^2 \varphi_d}{\partial x^2} + \frac{\partial^2 \varphi_j}{\partial z^2} - \frac{\partial^2 \varphi_r}{\partial x^2} \right] f'(x) + \lambda \frac{w^2}{\nu_m^2} \varphi_d - 2 \mu \left( \frac{\partial^2 \varphi_2}{\partial x^2} - \frac{\partial^2 \varphi_d}{\partial z^2} \right) = \rho^2 \left( k_x^2 f'(x) + \nu_m^2 \varphi_r \right).$$

(29)
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\[ K_m^2 = \frac{w^2}{v_m^2} - k_l^2, \quad m = d, s, \]

\[ K_s^2 = \frac{w^2}{v_s^2} - k_l^2. \]

Substituting this expressions (31) in the partial differential equations (27), (28) and (29) we find infinite systems of equations in the unknown series coefficients, expressed in (31) by \( M_l \).

In order to solve the infinite system we choose \( f(x) \) sinusoidal and triangular. For the infinite system we calculated a finite number of terms, taking into account energy conservation of the total reflected and transmitted field. Remark also that the intensities of the diffracted orders were normed against \( I_0 \), being the reflected intensity for a smooth surface.

In a first case we show how a normal incident ultrasonic beam is reflected by a sinusoidal and triangular curved surface.

It is seen that for both cases a minimum is realised for \( \lambda_m \equiv \lambda \) or \( v_m \equiv v/\lambda \). As for those frequencies the diffracted orders \(+1\) and \(-1\) propagate along the x-axis(surface) we calculate the corresponding velocity in water and found an approximate agreement (the calculated velocity was a little bit higher than the velocity of ultrasound of this frequency in water). In this way surface waves has been associated with this minimum and other minima from which surface waves could be associated in the solid confirm this theory.

In next picture we make a comparison between the reflection on steel and this on plexiglass. From Figure 6 it is evident that the influence of the elastic properties of the underlying media is much higher than the form of the curved surface.

Another interesting example was the study of reflection on polystyrene, where the transverse waves have a velocity of 1.250 m/sec. which is less than the velocity of ultrasound in water, and for which it is impossible to generate surface waves by oblique incidence. On next Figure 6 we calculated the reflected beam, and found minima at 1.003 and 1.672 MHz in polystyrene and surface waves in the water.

3. Bragg-angles

From light-diffraction by ultrasound we know that for certain incident angles most energy is diffracted in one order. Those angles are called Bragg angles.
Analogous results appear from the diffraction of ultrasound on periodic rough surfaces. Considering the relation between the incident angle $\theta_i$ and the diffraction angles $\theta_d$: 

$$\sin \theta_i - \sin \theta_d = (n-1) \lambda / \Lambda,$$  

we deduce those incident angles for which one diffracted order comes back in the incident direction. This is given by: 

$$\sin \theta_i = (n-k) \lambda / 2 \Lambda$$  

and we call them Bragg-angles.

In the special case $\lambda = \Lambda$ we found that depending on the level $h$ most reflected energy comes back in the direction $-30^\circ$. Taking $\lambda = 370 \mu m$ while $\Lambda = 350 \mu m$, and Bragg-angle $31.9^\circ$ confirmation of above statement is found.

Looking for a correlation between $h$ and the reflected intensity under Bragg angles we found in the case $\lambda = \Lambda$ and for the zero order in the cases water/steel and water/plexiglass the expression: 

$$|R_0|^2 = J_0^2 \left(2 \pi h / \lambda \right),$$  

found by L. M. Cherkashina [8] is followed for plexiglass/water, and that the characteristics of the medium (density, Lamé constant) are much more important than the form of the corrugation.

In Figure 8 we make another comparison between the different materials, in looking to the first order $|R_{-1}|^2$ diffraction in a Bragg angle incidence. Again the prediction by [8], for pure reflecting surfaces, given by:

$$|R_{-1}|^2 = J_1^2 \left(2 \pi h / \lambda \right),$$

is never followed.

4. Conclusion

Determination of the height of periodic irregularities at liquid/solid interfaces, through intensity measurements, can be realised. The necessity of theories and experiments other than on « rigid » or « pressure release » surfaces is clear, since influence of the material characteristics is considerable on quantitative results.
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Fig. 7. — $I_0$ et $I_{-1}$ versus angle of incidence at sinusoidal ($A=350\ \mu m$) water/stainless steel interfaces with $h=10\ \mu m$ (+), $h=30\ \mu m$ (□), and $h=50\ \mu m$ (△).

REFERENCES


Fig. 8. — $|R_{-1}|^2$ as function of the height $A=\lambda = 350\ \mu m$ and $0-30^\circ$ for water/steel (○), water/brass (△); water/ice (□) and water/plexiglass (+).
APPLICATION DE L'APPROXIMATION PARABOLIQUE A L'ACOUSTIQUE SOUS-MARINE.
Thèse de Doctorat de 3e cycle soutenue le 18 février 1985 par M. Bruno GRANDVUILLEMIN à l'Université d'AIX-Marseille-II.

On se propose de résoudre l'équation de Helmholtz au moyen de l'approximation parabolique dans un milieu de propagation bidimensionnel.

On explicite la solution pour un développement quadratique de l'opérateur pseudo différentiel de la propagation.

La résolution numérique est donnée par un algorithme aux différences finies utilisant un schéma Crank-Nicholson.