Stokes eigenmodes, potential vector – vorticity correlation and corner vortices in trihedral rectangular corner

E. Leriche & G. Labrosse

Laboratoire de Mathématiques de l’Université de Saint-Étienne, Faculté des Sciences et Techniques, Université Jean-Monnet, rue du Docteur Paul Michelson 23, Saint-Étienne F-42023, France
(emmanuel.leriche@univ-st-etienne.fr).

Université Paris-Sud, LIMSI-CNRS, Bâtiment 508, BP 133, 91403 Orsay Cedex, France
(gerard.labrosse@limsi.fr)

Abstract :
A first deep insight of the Stokes eigenmodes in the fully confined domain (square and cubical) is reported. First, the dynamics of the Stokes eigenmodes flows is established leading to a collinear relationship between the vector potential and the vorticity. Secondly, from the analysis of the corner eddies made by Moffatt, the existence of corner vortices in a fully confined trihedral rectangular corner is reported when numerically computing the Stokes eigenmodes in the cube. A first analysis in term of types, sequence, and flow structures is provided.

Key-words :
Stokes flow; Moffatt’s corner eddies; 3D corner eddies

1 Introduction

The knowledge of Stokes eigenmodes in a square/cube (the simplest two- and three-dimensional confined domain) or in any bounded domain could provide some insight into the understanding or analysis of turbulent instantaneous flow field in a geometry as simple as, for instance, the driven cavity. Stokes eigenmodes are not analytically known except when they are periodic in all, or in all but one, space directions. If they are indeed constrained to verify velocity no-slip conditions on a closed boundary they can only be determined by numerical approach. The aim of the present contribution is to provide a first deep insight into the Stokes eigenspace in the most simple confined geometries, namely a square or a cube, and then to provide some generalisation to any bounded domain. Owing to the lack of knowledge of the Stokes spectrum, except from the theoretical asymptotic predictions, $\lambda_k \simeq k^{2/d} + O(k^{1/d})$ for a $d$-D problem, proposed by Constantin and Foias ([2]), an accurate computation of a significant part of the spectrum is needed to determining the coefficients of the afore mentioned theoretical predictions (see the analysis limited to the square case published in [6] and preliminary results for the cubical case spectrum in [10]). First, the unsteady dynamics of any Stokes flows is settled leading to a general exact vectorial relationship between the vector potential and the vorticity. Then, this relation is applied to the Stokes eigenmodes confined in the disk, in the plane channel, in the square and in the cube. It shows that, in the core of these domains, and therefore of any closed 2D or 3D domain, these modes verify a collinear relationship between the vector potential and the vorticity. This is a definitive answer to the discussion opened by Batchelor in the sixties about the existence of such a relation for 2D inviscid (large Reynolds number) steady laminar flows or for 2D viscous flows (zero Reynolds number). Secondly, guided by the analysis of the Stokes eigenmodes in the square where Moffatt’s corner eddies are occurring in an infinite sequence of decreasing intensities and enjoying several symmetry properties, the existence of
corner vortices in a fully confined trihedral rectangular corner is reported when numerically computing the Stokes eigenmodes in the cube. Like in the square, these corner vortices bear some symmetry properties and the 3D corner flow structure is identified. Over the last years, analytical description of 3D corner eddies have regained interest in the literature and have been reported like, for instance, in the cone where these 3D corner eddies are somehow the natural extension of the 2D Moffatt’s ones [14, 3, 12]. To the author’s knowledge, such an analytical description in a fully confined trihedral rectangular corner is still missing in the literature and a first analysis of two corner vortices obtained from two Stokes eigenmodes in the cube is provided.

2 The Stokes eigenproblem governing equations

Let us write the three-dimensional Stokes eigenproblem with primitive variables, in the open domain \( \Omega = ]-1,1[^d \) with \( d = 2,3 \) and with coordinates \( x = (x, y) \) or \( x, y, z \), the eigenvalues \( \lambda \) (strictly negative ([15, 2]), corresponding to the algebraic temporal growth rate of \( \vec{v} \)), where \( \vec{v} = (u, v) \) or \( (u, v, w) \) is the velocity field and \( p \) the pressure. We denote the closure of \( \Omega \) by \( \overline{\Omega} \) and the boundary by \( \partial \Omega \). Homegeneous Dirichlet boundary conditions are imposed on the velocity. The eigensystem reads as follows:

\[
\begin{align*}
\lambda \vec{v} &= \vec{\nabla}^2 \vec{v} - \vec{\nabla} p \quad \text{for } x \in \Omega, \\
\vec{\nabla} \cdot \vec{v} &= 0 \quad \text{for } x \in \Omega, \\
\vec{v} &= 0 \quad \text{for } x \in \partial \Omega,
\end{align*}
\]

The alternative form of the system (1) is its vector potential formulation, with \( \vec{\psi} \) defined by \( \vec{v} = \vec{\nabla} \times \vec{\psi} \) and verifying

\[
\begin{align*}
(\lambda - \vec{\nabla}^2) \vec{\nabla}^2 \vec{\psi} &= 0 \quad \text{for } x \in \Omega, \\
\vec{\nabla} \cdot \vec{\psi} &= 0 \quad \text{for } x \in \Omega,
\end{align*}
\]

together with homogeneous no-slip/no-flux boundary conditions ([4]),

\[
\vec{\psi} = \frac{\partial \vec{\psi}}{\partial n} = 0 \quad \text{for } x \in \partial \Omega,
\]

\( n \) being the coordinate evaluated along \( \vec{n} \), the unit vector normal to \( \partial \Omega \).

3 Stokes Solvers

Computing the Stokes eigenmodes can be made from either their (velocity-pressure) primitive variables, Eqs. (1), or vector potential formulation, Eqs. (2-4). It has to be noticed that extending the 2-D velocity-pressure formulation into 3-D (see Eq. (1)) is straightforward while extending the stream function to the vector potential formulations leads to a new system devoid of any Lagrange multiplier associated with enforcing the divergence constraint, Eq. (3). Designing and implementing a numerical treatment of such a problem requires an important investment: defining divergence free trial functions fulfilling two boundary conditions on each wall, and spectrally solving an eigenvalue problem of a fourth order differential operator is a non trivial task. For this reason, we have opted for the primitive variables formulation given by Eqs (1), and the velocity-pressure uncoupling is made by the Projection-Diffusion (PrDi) algorithm ([5, 11]). This scheme is known to be consistent with the continuous uncoupled problem.
and is one of the two solvers involved in the computation of the Stokes eigenmodes in the square ([6, 7]). Its extension to the cubical domain is straightforward [10]. Section 3.2 in [5] provides all the details about the discrete formulation of the problem. The spatial discretization is based on the usual Chebyshev Gauss-Lobatto spectral collocation method ([1]). Let \( N \) be the polynomial cut-off frequency, where \( N + 1 \) is the number of collocation points, in each space direction. Numerically determining the 3-D Stokes eigenmodes, even in the simple cubical domain, is not an easy task, due to, both, the algorithmic conception and the to-day machine efficiency [10]. It leads to handle a huge eigenproblem to be solved. Regarding this last aspect, the authors believe that it is at the frontier of the nowadays eigensolver capabilities.

4 Stokes eigenmodes dynamics

In [8], the unsteady dynamics of the Stokes flows, where \( \nabla^2 p = 0 \), is shown to verify the vector potential - vorticity (\( \vec{\psi}, \vec{\omega} \)) correlation
\[
\frac{\partial \vec{\omega}}{\partial t} + \vec{\omega} + \vec{\Pi} = 0,
\]
where the field \( \vec{\Pi} \) is the pressure-gradient vector potential defined by \( \vec{\nabla} p = \vec{\nabla} \times \vec{\Pi} \). This correlation is analyzed for the Stokes eigenmodes, \( \frac{\partial \vec{\omega}}{\partial t} = \lambda \vec{\psi} \), subjected to satisfy no-slip boundary conditions on any 2D closed contour or 3D surface. The effect of \( \vec{\Pi} \) damps progressively moving inwards the domain from the boundary, and it is established that an asymptotic linear relationship appears, verified in the core part of the domain, between the vector potential and vorticity, \( (\vec{\omega} - \vec{\omega}_0) = -\lambda \vec{\psi}, \vec{\omega}_0 \) being a constant offset field, possibly zero. In [8], this relation is highlighted in examples like the Stokes eigenmodes confined in the disk, in the plane channel, in the square [6] and in the cube.

5 About the 2D corner Moffatt’s eddies

As it is well known since Moffatt’s work ([13]), the Stokes eigenmodes in a square contain an infinite sequence of similar corner eddies, singular at each of the square four corners, verifying \( \Delta^2 \psi = 0 \) with \( \psi = \frac{\partial \psi}{\partial n} = 0 \) on the boundaries. They are not specific of the Stokes eigenmodes but of their symmetry properties: the corner eddies are even about the square diagonals (Figs. 1(a)), odd (Figs. 1(b)), and without diagonal symmetry for (Figs. 1(c)). The intensity of each eddy decreases exponentially towards the corners. This reduction is much more rapid when the eddies are odd about the square diagonals. As a consequence, these eddies looks mainly even in the family without diagonal symmetry in the Figs. 1(c). Their complete description is supplied in [7] providing their main characteristics (intensities and position of the corner eddies centers in each symmetry case).
6 Stokes eigenmode in the cube

6.1 Visualization of Stokes eigenmode in the cube

As a matter of preliminarily illustrating the spatial structure of one Stokes eigenmode in the cube, Fig. 2 shows the one mode of the single states (eigenvalue with a single multiplicity, \( \lambda = -45.366 \)), as 3-D visualizations of 4 iso-surfaces. For each component of the primitive variables \((\vec{v}, p)\), of the vector potential \(\psi\), and of the vorticity \(\vec{\omega} = \vec{\nabla} \times \vec{v}\), two sets of opposite values are selected for the iso-surfaces. Obvious symmetry properties are exhibited, requiring a deeper analysis that is out of the scope of this paper [9]. It allows to qualitatively verify the relationship between the vector potential \(\psi\) and vorticity \(\omega_0\), see section 4.
6.2 About the 3D corner eddies

In contrast with the odd/even Moffatt eddies structure in 2D corner (not to mention their sym-
metry properties), the situation appears to be much more rich in term of types, sequence, and
flow structures of the eddies in 3D trihedral rectangular corner. The figures 3 and 4 shows –
with several points of vue and zooms– the streamlines (colored by the velocity norm intensity)
in the corner of two Stokes eigenmodes (obtained by the $N = 64 \, \text{PrDi}$ solver) with eigenvalue
$\lambda = -36.680$ and $-45.366$, both eigenvalues with single multiplicity. To the author’s knowl-
dge, appart from the first attent to analytically determine Stokes flow in a trihedral rectangular
corner [3], there is no clear view on what are the 3D corner eddies in such a corner. In contrast
with the eddies structure in 2D corner, the figures 3 and 4 suggest that the flow structures of
the eddies is connected with the edges or even with the core flow of the eigenmodes. This is
rather different type of flow structure regarding the one developping in the cone. These 3D
corner eddies in the cone are to a certain extend the natural extension of the 2D Moffatt’s ones
[14, 3, 12].

Figure 3: Streamlines (colored by the velocity norm intensity) in the corner of the Stokes eigenmodes
(obtained by the $N = 64 \, \text{PrDi}$ 3D solver) in a cube with eigenvalue $\lambda = -36.680$ of a single multiplicity.

Figure 4: Streamlines (colored by the velocity norm intensity) in the corner of the Stokes eigenmodes
(obtained by the $N = 64 \, \text{PrDi}$ 3D solver) in a cube with eigenvalue $\lambda = -45.366$ of a single multiplicity.

7 Conclusions

The Stokes eigenmodes in the most simple fully confined domains (square and cube) are numer-
ically computed and analyzed. It leads to establish a collinear relationship between the vector
potential and the vorticity, relation which is verified for every Stokes eigenmodes not only in
the square or in the cube, but in any confined domain with boundaries of arbitrary shape. From
two computed Stokes eigenmodes in a cunbe, a preliminary analysis of the corner vortices in a
fully confined trihedral rectangular corner is then provided. They do not appear to be a natural
extension of the 2D corner Moffatt’s eddies.

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References


