Four channel Wigner-Smith matrix formalism applied to the scattering by a fluid layer embedded in semi infinite solids

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Abstract:

The acoustic scattering by a fluid slab between two semi infinite solid media is revisited from the point of view of a four channel resonant scattering formalism. For a plane and monochromatic longitudinal (P=L) or transversal (P=T) wave incident from each solid, the reflection coefficients \( r^P \) and transmission coefficients \( t^Q \) by the fluid layer (\( Q=L, T \) represents the polarization of the scattered waves) are the components of a 4x4 symmetric, unitary scattering matrix \( S \) (\( S^4 = S^T = 1 \)). In the particular case of identical solids, it is shown that \( S \) has to be written as the product of 2 unitary scattering matrices: \( S^{D\prime} \) - corresponding to the scattering by the vacuumed layer-, and \( S^{\prime\prime}\) - denoting the pure resonant part of \( S \). The Wigner-Smith matrix \( Q_y = -j(\partial_y S)S^T \) is analyzed (\( \partial_y \) is the partial derivative with respect to a given input parameter \( x_{1,2,3,4} = f, c_L, c_T, c_F \)), this formalism being the multichannel extension of Phase Gradient Method.

Résumé:

L’étude de la diffusion acoustique par un slab fluide entre deux solides homogènes est réexaminée dans le cadre du formalisme de la matrice \( S \) de diffusion résonnante à 4 canaux. Considérant une onde incidente plane monochromatique longitudinale (P=L) ou transversale (P=T), la matrice de diffusion \( S \) (4x4) symétrique unitaire est élaborée à partir des coefficients de réflexion \( r^Q \) et de transmission coefficients \( t^Q \) par la couche fluide (\( Q=L, T \) représente la polarisation des ondes diffusées). Dans le cas particulier de solides identiques, on montre que \( S \) s’écrit comme produit de 2 matrices de diffusion unitaires: \( S^{D\prime} \) - matrice de diffusion de la couche vide-, and \( S^{\prime\prime} \)-contribution purement résonnante de \( S \). Le formalisme de la matrice de Wigner-Smith \( Q_y = -j(\partial_y S)S^T \) est alors mis en place (\( \partial_y \) est la dérivée partielle par rapport à l’un des paramètres d’entrée \( x_{1,2,3,4} = f, c_L, c_T, c_F \)), afin de généraliser la Méthode des Gradients de Phase à une situation de diffusion multicanal.

Keywords:

Multichannel Scattering ; Phase Gradient ; Wigner-Smith Matrices

1 Construction of the \( S \) matrix

The present work is intended to prepare studies on effective medium approximations of a slab region randomly filled with scatterers in a solid. Let consider the simple problem of the scattering by a fluid layer \( F_2 \) (thickness \( d \), density \( \rho_F \), phase velocity \( c_F \)) between two semi infinite solids \( S_1 \) and \( S_3 \) (density \( \rho_{S,m} \), longitudinal and transversal phase velocities \( c_{L,m} \) and \( c_{T,m} \)).
As for a plane and monochromatic incident wave (frequency \(f\), incidence angle \(\theta\)) from each solid \(S_m\) \((m=1,3)\), two elementary polarizations \(P=\text{L}\) (for longitudinal, \(T\) (for transversal) are available, such a study requires the multichannel framework provided by the scattering matrix \(S\) formalism.

Assuming the continuity of the normal and tangential stress components and the normal displacement component at the boundaries \(z=\pm d/2\), the reflection \(r_{m}^{PQ}\) and transmission \(t_{m}^{PQ}\) coefficients of the structure are needed to build the \(S\) matrix. Their expression are (1)

\[
\begin{align*}
    r_{m}^{LL} &= \frac{(C_{A,m}^{+} + i\tau_{m})(C_{S,m}^{+} - i\tau_{m}) + (C_{A,m}^{-} + i\tau_{m})(C_{S,m}^{-} - i\tau_{m})}{D}, \\
    r_{m}^{LT} &= -4k_{m}k_{T,m}(2k_{T,m}^{2} - k_{T,m}^{2})\left[T(C_{S,m}^{-} - i\tau_{m}) + C(C_{A,m}^{+} + i\tau_{m})\right]/D, \\
    r_{m}^{TL} &= \frac{(C_{A,m}^{+} - i\tau_{m})(C_{S,m}^{+} - i\tau_{m}) + (C_{A,m}^{-} + i\tau_{m})(C_{S,m}^{-} + i\tau_{m})}{D}, \\
    r_{m}^{TT} &= \frac{(C_{A,m}^{+} + i\tau_{m})(C_{S,m}^{+} + i\tau_{m}) + (C_{A,m}^{-} + i\tau_{m})(C_{S,m}^{-} - i\tau_{m})}{D},
\end{align*}
\]

where (2)

\[
\begin{align*}
    D &= (C_{A,m}^{+} + i\tau_{m})(C_{S,m}^{+} - i\tau_{m}) + (C_{A,m}^{-} + i\tau_{m})(C_{S,m}^{-} - i\tau_{m}), \\
    C_{A,m}^{\pm} &= R_{m}^{T}C, \quad C_{S,m}^{\pm} = R_{m}^{S}C, \quad \tau_{m} = (\rho_{p}/\rho_{f})(k_{z,m}/k_{T,m}), \\
    R_{m}^{T} &= 4k_{m}k_{T,m}^{2}k_{T,m}^{2} + (2k_{T,m}^{2} - k_{T,m}^{2})^{2}, \quad T = C^{-1} = \tan(\pi f k_{T,m} d), \\
    \alpha_{m} &= -(i)2k_{m}^{2}(k_{z,m}/k_{T,m})(2k_{T,m}^{2} - k_{T,m}^{2}).
\end{align*}
\]

The indexes \(m=1,3\) and \(n \neq m\) stand for the incidence solid medium and the second solid medium respectively. Denoting by \(\theta_{L,m}\) and \(\theta_{T,m}\) the longitudinal and transversal incidence angles respectively and by \(\theta_{f}\) the refraction angle in the fluid, the projections of the involved normalized wave vectors \((k_{L,m} = 1/c_{L,m}, k_{T,m} = 1/c_{T,m}, k_{f} = 1/c_{f})\) are

\[
\begin{align*}
    k_{s} &= \sin\theta_{L,m}/c_{L,m} = \sin\theta_{T,m}/c_{T,m} = \sin\theta_{f}/c_{f}, \quad \text{along the x-axis}, \\
    k_{z,m} &= \cos\theta_{L,m}/c_{L,m}, \quad k_{z,T,m} = \cos\theta_{T,m}/c_{T,m}, \quad \text{and} \quad k_{zf} = \cos\theta_{f}/c_{f}, \quad \text{along the z-axis}. 
\end{align*}
\]

In the symmetric case of two identical solid media surrounding the slab \((S_{3} \equiv S_{1})\), the four channel scattering matrix \(S\) built up from the conservation of the energy flow vector perpendicularly to the interfaces (Franklin et al. (2001)) is.
with the normalization coefficient \( \beta = (k_{T,1}/k_{L,1})^{1/2} \). \( \mathbf{S} \) is a symmetric and unitary matrix \( \mathbf{S}^{\dagger} \mathbf{S} = \mathbf{S} \mathbf{S}^{\dagger} = \mathbf{I} \), where \( \dagger \) is the adjoint operator. In the following, the study will be limited to the case of light fluids \( (c_{F} < c_{T,1} < c_{L,1}) \), and the different graphics will be plotted for a water slab \( F_{2} \) (thickness \( d = 5 \) mm, density \( \rho_{F} = 1000 \) kg.m\(^{-3}\), phase velocity \( c_{F} = 1470 \) m.s\(^{-1}\)) in aluminum solid \( S_{1} \) (density \( \rho_{S,1} = 2790 \) kg.m\(^{-3}\), longitudinal and transversal phase velocities \( c_{L,1} = 6380 \) m.s\(^{-1}\) and \( c_{T,1} = 3100 \) m.s\(^{-1}\)).

2 Analysis of the \( \mathbf{S} \) matrix: resonant and background scattering.

The \( \mathbf{S} \) matrix formalism offers a powerful tool to break apart scattering mechanisms into simple and understandable contributions. As the analytical determination of the \( \mathbf{S} \) eigenvalues from the characteristic equation \( \det(\mathbf{S} - \lambda \mathbf{I}) = 0 \) is difficult, a way to obtain these quantities consists in the study of the \((2 \times 2)\)-subminors of the determinant \( \det(\mathbf{T}) \) of the transition matrix \( \mathbf{T} = (\mathbf{S} - \mathbf{I})/(2j) \). This provides the remarkable identities written as follows:

\[
\begin{align*}
(t_{1}^{LL} - 1)/\beta t_{1}^{LT} &= -t_{1}^{LL}/\beta t_{1}^{LT} = -\beta r_{1}^{LT}(t_{1}^{TT} + 1) = -\beta t_{1}^{LT}/t_{1}^{TT} .
\end{align*}
\]

The use of the previous relations enables to factorize easily the characteristic polynomial \( \det(\mathbf{S} - \lambda \mathbf{I}) \). Finally, the four unitary eigenvalues of \( \mathbf{S} \) are \( \lambda = \{ S_{S}, S_{A}, 1, 1 \} \) where

\[
S_{S/A} = r_{1}^{LL} - t_{1}^{TT} \mp (t_{1}^{LL} + t_{1}^{TT}) = -(C_{S/A} \mp j\tau_{1})/(C_{S/A} \mp j\tau_{1}) ,
\]

The open eigenchannels -associated with the eigenvalues \( S_{S} \) and \( S_{A} \)- correspond respectively to symmetric-longitudinal and antisymmetric-longitudinal polarizations. The eigenchannels associated with the double eigenvalue equal to 1 are closed and should correspond respectively to symmetric-transversal and antisymmetric-transversal polarizations if these channels were open \( (i.e. \) in the case of a solid slab for example).

FIG. 2 – Plot in the \((f, \theta_{L})\)-plane of the non null eigenvalue moduli of the transition matrix (left: \( |T_{S}| \); right: \( |T_{A}| \)).
As a consequence, only two eigenvalues of the transition eigenmatrix \( T_{\text{eig}} = (S_{\text{eig}} - \textbf{I})/(2j) \) defined from the scattering eigenmatrix \( S_{\text{eig}} = \text{diag}\{S_S, S_A, 1, 1\} \) are non null and denoted as \( T_{S/A} = (S_{S/A} - \textbf{I})/(2j) \). The plots of their moduli in the (frequency, longitudinal incidence angle)-plane \((f, \theta_L)\) are given in FIG.2. They both exhibit antiresonances (in fact, resonances in counterphase with a constant transition amplitude component). Compared to other resonant scattering problems (Rembert et al. (2004)), this behavior can be easily explained by separating the whole scattering amplitudes as a combination of non resonant (or background) scattering terms and purely resonant scattering contributions. Mathematically speaking, the shape of the transition amplitudes is mainly due to the minus (-) sign that occurs in front of the non unimodular eigenvalues of \( S \) in Eq. (5). Indeed, when plotting the transition amplitudes \( T_{S/A}^{(*)} = (-S_{S/A} - \textbf{I})/(2j) \) instead of \( T_{S/A} \), one obtains a pure Breit-Wigner resonance spectrum (see FIG. 3). Indeed, at a given incidence angle \( \theta_L \), a first order expansion of \( T_{S/A} \) shows that each resonance frequency is close to a root of \( C_{S/A} = 0 \) and each resonance half-width is nearly equal to \( \pm \tau_{\pm}/(\partial C_{S/A}/\partial x) \). As a consequence, the scattering eigenmatrices \( S_{\text{eig}}^{(b)} = \text{diag}\{-1, -1, 1, 1\} \) and \( S_{\text{eig}}^{(*)} = \text{diag}\{-S_S, -S_A, 1, 1\} \) are defined so that \( S_{\text{eig}} = S_{\text{eig}}^{(b)} S_{\text{eig}}^{(*)} \).

\[
\begin{align*}
\begin{bmatrix}
T_{S}^{(*)} \\
T_{A}^{(*)}
\end{bmatrix},
\end{align*}
\]

FIG. 3 – Plot in the \((f, \theta_L)\)-plane of the resonant eigentransition amplitudes (left : \( |T_{S}^{(*)}| \); right : \( |T_{A}^{(*)}| \)). In order to connect these eigenmatrices with the original scattering matrix \( S \), the calculation of the eigenvectors is straightforwardly processed. Once normalized, the real rotation matrix \( \mathbf{R} \) involved in the change of basis relation, \( S = \mathbf{R} S_{\text{eig}} \mathbf{R}^\dagger \), is

\[
\mathbf{R} = \begin{bmatrix}
\gamma & -1 \\
1 & \gamma
\end{bmatrix} \begin{bmatrix}
1 & 1 & \gamma & \gamma \\
-1 & 1 & -\gamma & -\gamma \\
-\gamma & -\gamma & 1 & 1 \\
-\gamma & \gamma & -1 & 1
\end{bmatrix} 
\] \( \gamma = (R_{S}^* + R_{A}^*)/(R_{S}^* - R_{A}^*) \).

As they do not depend on the frequency, the components \( r_{11} \) and \( r_{13} \) of \( \mathbf{R} \) are plotted versus the longitudinal incidence angle \( \theta_L \) in FIG. 4.
Introducing the change of basis relations \( S^{(r)} = R S_{\text{eg}}^{(r)} R^t \) and \( S^{(b)} = R S_{\text{eg}}^{(b)} R^t \), the scattering matrix \( S \) is factorized as the product \( S = S^{(b)} S^{(r)} \). \( S^{(r)} \) describes the pure resonant scattering of the fluid slab, and \( S^{(b)} \) the non resonant scattering of the slab. More precisely, it is demonstrated that \( S^{(b)} \) is nothing but the \( S \) matrix written in the particular case \( \rho_p = 0 \), i.e. the scattering by a vacumed slab. In that case, the scattering coefficients related to the transmission terms vanish and the components of \( S^{(b)} \) only depend on the rotation matrix \( R \).

3 Wigner-Smith matrix

The time-delay or Wigner-Smith matrix \( Q_x \) is defined from the scattering matrix as

\[
Q_x = -j(\partial_x S) S^t. \quad (9)
\]

Recalling that \( S^{-1} = S^t \), \( Q_x \) is the generalized logarithmic derivative of \( S \) with respect to a parameter \( x \) involved in the expression of the scattering matrix elements. This matrix is self-adjoint (\( Q_x = Q_x^t \)) and its eigenvalues are pure phase partial derivatives with respect to \( x \), connected with the eigenphase derivatives of \( S \) (Franklin et al. (2006)). \( Q_x \) is the multichannel generalization of the Phase Gradient Method used to investigate the resonant phase of scattering coefficients (Lenoir, et al. 2003). Considering the particular set of parameters \( x_{i+4,4} = (f, c_{xL}, c_{Lx}, c_{Rx}) \), it is shown, as in the cases of one channel and two channel scattering (Lenoir, et al. 2003, Franklin et al. 2006), that the remarkable equation

\[
\sum_{i} x_i Q_{x_i} = 0, \quad (10)
\]

is still verified. When written at a resonance frequency of the scatterer, this relation represents a reactive power balance. Another interest of the \( Q_x \) formalism lies obviously in the linearization of each possible factorization of \( S \). Inserting the change of basis relation \( S = R S_{\text{eg}} R^t \) in definition (9) gives

\[
Q_x = Q_x^{(r)} + Q_x^{(p)}. \quad (11)
\]

In the previous relation, \( Q_x^{(r)} = -jR (\partial_x S_{\text{eg}}^{(r)}) S_{\text{eg}}^{(r)\dagger} R \) is the purely resonant part of \( Q_x \) related to \( S^{(r)} \); its eigenmatrix is \( (\partial_x S_{\text{eg}}^{(r)}) S_{\text{eg}}^{(r)\dagger} = \text{diag}\left\{ -2\partial_x \delta_S^{(r)}, -2\partial_x \delta_S^{(r)}, 0, 0 \right\} \), where \( -\delta_S^{(r)} \) are the half resonant eigenphases of \( S \) (i.e. \( S_{\delta_S} = -\exp(-j2\delta_S^{(r)}) \)), the trace of \( Q_x^{(r)} \) is a pure Breit-Wigner resonance spectrum (each resonance peak height are twice the inverse of their width). \( Q_x^{(p)} \) is
the potential contribution of $Q_{x}$, but slightly different from the background scattering due to $S^{(b)}$. Indeed, its eigenmatrix $Q_{x}^{(p)}$ depends on the half resonant eigenphases:

$$Q_{x}^{(p)} = \text{diag}\{\cos(\delta_{x}^{(v)}),\cos(\delta_{y}^{(v)}),-\cos(\delta_{y}^{(v)}),-\cos(\delta_{z}^{(v)})\} \times \partial_{x} \gamma / (1 + \gamma^{2}) .$$

(12)

$Q_{x}^{(p)}$ is a null trace matrix, proving that $\text{tr}(Q_{x}^{(p)}) = \text{tr}(Q_{x}^{(t)})$ contains the whole resonant information about the scatterer. In the particular case $x = f$, $Q_{x}^{(p)}$ is null ($\gamma$ is independent of $f$) and then, $Q_{f} = Q_{x}^{(p)}$. In Fig. 5, the first symmetric eigenvalues of $Q_{x}^{(t)}$ and $Q_{x}^{(p)}$ are plotted in the $(f, \theta_{L})$-plane: their respective amplitudes clearly indicate that the role of the potential scattering is negligible compared to the resonant scattering in $Q_{x}^{(p)}$.

![Fig. 5 – Plots in the $(f, \theta_{L})$-plane of the symmetric eigenvalues of $Q_{x}^{(t)}$ (left), and $Q_{x}^{(p)}$ (right).](image)

5 Conclusive remarks

The Wigner-Smith matrix $Q$ formalism is a natural filtering tool to isolate resonant contributions in a many scattering channel situation. This ability concerns, not only the frequency variable, but all the independent input parameters of the problem. The $S$ matrix formalism is used as a “preprocessing engine” in order to exploit fully the $Q$ matrix properties.

References


