Nonlinear excitation of low-frequency equatorial waves

Reznik Grigory, Zeitlin Vladimir

P.P. Shirshov Institute of Oceanology
Geophysical Fluid Dynamics Laboratory
36 Nakhimovskiy prospect, 117997 Moscow, Russia
greznik11@yahoo.com

Abstract:

We study non-linear interactions between the waves trapped in equatorial waveguide and the barotropic Rossby waves freely propagating across the waveguide. The model of two-layer shallow water on the equatorial beta-plane is used. We consider the interaction of one barotropic Rossby wave with two equatorial modes, and with one equatorial mode in the presence of zonal equatorial current. It is found in both cases that the free barotropic Rossby wave resonantly excites the baroclinic equatorial Rossby or Yanai waves with amplitudes much exceeding its own amplitude. The baroclinic wave envelopes obey Ginzburg-Landau type equations and exhibit non-linear saturation and formation of characteristic “domain wall” and “dark soliton” defects. In turn, self-interaction of the “trapped” part of flow results in a non-linear scattering of the free wave on the waveguide. Thus, this mechanism provides both generation of large-amplitude equatorial waves and back reaction of the equatorial region on the mid-latitudes.

Résumé :


Key-words : equatorial waveguide; barotropic Rossby waves; nonlinear interactions

1 Introduction

There is a special narrow domain in the vicinity of the Earth equator in the ocean and atmosphere – the so-called equatorial waveguide. The waves trapped in the waveguide (Gill, 1982) and propagating along it, play an important role in the dynamical processes which determine the Earth climate, like El Nino phenomenon (cf. Delcroix et al, 1991; Boulanger, Menkes, 1995) or tropospheric Madden-Julian oscillations (Hendon, Salby, 1994). At the same time, the non-trapped barotropic Rossby waves can freely propagate across the equator and the equatorial waveguide is transparent for them. Here we study non-linear interaction between the trapped equatorial and the free non-trapped barotropic Rossby waves. Such interaction, first, turns out to be an effective source of the low-frequency equatorial waves, and, second, it is able to provide the observed correlation between the large-scale processes at mid-latitudes and
equatorial regions, c.f. Hoskins, Yang G.-Y. (2000). At the same time, this consideration can be useful for other situations where the interactions between the free and the trapped modes are possible, for example, for edge waves near the shores or for the topographic trapped modes.

2 Model

We use a simple model where the atmosphere or the ocean are represented by two-layer fluid contained between a rigid lid and a rigid flat bottom on the so-called equatorial beta-plane (Gill, 1982). The model equations written in the shallow water approximation have the form (Benilov, Reznik, 1996):

\[
\begin{align*}
\frac{\partial \mathbf{u}_i}{\partial t} + \nabla \cdot (\mathbf{u}_i \mathbf{v}) + p_i + \beta \hat{y} \times \mathbf{u}_i &= 0, \quad i = 1, 2, \quad (1a) \\
h_{ii} + \nabla \cdot (h_i \mathbf{u}_i) &= 0, \quad i = 1, 2. \quad (1b)
\end{align*}
\]

Here \(x, y, z\) axes are directed eastward, northward, and upward, respectively, \(h_i\) are the depth layers, \(p_i\) are the horizontal velocities, \(p_i\) - the pressures, \(p_2 = p_1 - g' h_1\); \(\rho_i\) - the densities in the layers, \(g' = g(\rho_2 - \rho_1)/\rho_1\) is the reduced gravity, \(\beta\) is the parameter characterizing the joint influence of the Earth rotation and sphericity, \(\hat{z}\) is the vertical ort.

In terms of the barotropic streamfunction \(\psi\), \(h_1 \mathbf{u}_1 + h_2 \mathbf{u}_2 = H \hat{z} \times \nabla \psi\), baroclinic velocity \(\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2\) and the upper layer depth \(h\) the non-dimensional system (1) becomes:

\[
\begin{align*}
\nabla^2 \psi_x + \psi_x &= \varepsilon N_\psi (\psi, \mathbf{u}, h), \\
\nabla^2 \psi_y + \psi_y + \nabla h + y \hat{z} \times \mathbf{u} &= \varepsilon N_u (\psi, \mathbf{u}, h), \\
h_x + u_x + v_y &= \varepsilon N_h (\psi, \mathbf{u}, h), \quad \varepsilon = \varepsilon_0.
\end{align*}
\]

The linear wave spectrum of the model consists of the barotropic Rossby waves:

\[
\psi_0 = A_\psi e^{i(\theta_0 + \nu_0)} + c.c., \quad \theta = k x - \sigma t, \quad \sigma = -k \sqrt{k^2 + l^2}, \quad (4)
\]

and the baroclinic equatorial waves

\[
(\mathbf{u}_0, h_0) = (U, H) A e^{i\hat{\theta}} + c.c., \quad \hat{\theta} = \hat{k} x - \hat{\sigma} t, \quad \hat{\sigma}^3 - (\hat{k}^2 + 2m + 1)\hat{\sigma} - \hat{k} = 0. \quad (5)
\]

Here \(k, \sigma\) and \(\hat{k}, \hat{\sigma}\) are the zonal wavenumber and frequency of the barotropic and equatorial waves, respectively; the wave amplitudes \(A, A_\psi\) are constant in the linear approximation; the integer \(m = 1, 2, ...\) is the equatorial wave meridional wavenumber. The amplitudes \(U = (U, V), H\) decay rapidly away from the equator:

\[
\begin{align*}
U &= \frac{\hat{\sigma} \varphi - \hat{k} \varphi'}{\hat{\sigma}^2 - \hat{k}^2}, \quad H = \frac{\hat{k} \varphi - \hat{\sigma} \varphi'}{\hat{\sigma}^2 - \hat{k}^2}, \quad V = \varphi = \hat{H}_m(y) e^{-y^2/2} \sqrt{\frac{2^m m! \sqrt{\pi}}{\sqrt{2^m m!}}} \quad (6a, b, c)
\end{align*}
\]

Here \(\hat{H}_m(y)\) - Hermite polynomial of \(m\) -th order. We are interested in the low-frequency equatorial baroclinic Rossby and Yanai waves with frequencies \(\hat{\sigma} < 1\).

3 Triple interactions

Because of special form of the non-linear terms in (2) only the interaction between one barotropic and two equatorial waves is possible. There are two types of synchronism conditions:
As usual, the solution to (2), (3) is sought in the form of multi-timescale asymptotic expansions

$$(\psi, u, v, h) = (\psi_0, u_0, v_0, h_0)(x, y, t, T) + \varepsilon(\psi_1, u_1, v_1, h_1)(x, y, t, T) + \ldots,$$

where $T = \sigma$ is the slow time and the lowest-order fields $\psi_0, u_0, v_0, h_0$ are represented as linear waves (4), (5) with amplitudes depending on $T$. The key point is that the interactions between the trapped baroclinic waves are not able to generate resonances in the r.h.s. of (2a) and the barotropic amplitude $A_\psi$ can be taken constant. Equations for amplitudes $A_1, A_2$ of baroclinic waves are reduced to the simple form:

$$\partial^2 A_i / \partial T^2 = K^{(\pm)} |A_\psi|^2 A_i, \quad i = 1, 2,$$

where $K^{(\pm)}$ is a constant real coefficient depending on the structure and parameters of interacting waves, and the superscript $\pm$ corresponds to the signs in (7a,b). The coefficient $K^{(+)\pm}$ is positive, i.e. the barotropic Rossby mode excites exponentially growing baroclinic equatorial modes with frequencies smaller than its own frequency. At the same time $K^{(-)\pm}$ is negative, i.e. the interaction results only in slow oscillations of the baroclinic modes amplitudes if the barotropic frequency lies between the baroclinic ones as in case (7b). In what follows we concentrate on the most interesting case (7a).

Equation (9) is valid until the exponential growth causes the baroclinic amplitudes $A_i$ to be $O(\sqrt{\varepsilon})$ so that the non-linear term $\varepsilon N_\psi$ in (2a) becomes of the order of unity. At this stage the secondary barotropic flow $\varepsilon \psi_1$ determined by the equations

$$\nabla^2 \psi_{1T} + \psi_{1x} = -s(\partial_{xx} - \partial_{yy})(u_0v_0) + s\partial_{xy}(u_0^2 - v_0^2), \quad \psi_1|_{t=0} = 0,$$

becomes comparable with the primary wave $\psi_0$. Beyond this threshold interaction of the correction $\varepsilon \psi_1$ with the baroclinic modes arrests the growth of the baroclinic amplitudes and influence of the baroclinic waves on the barotropic one cannot be neglected. To study the eventual saturation the asymptotic expansion (8) should be rearranged as follows:

$$\psi = \psi_0(x, y, t, T_1, T_2) + \varepsilon\psi_1(x, y, t, T_1, T_2) + \ldots,$$

with the slow times $T_n = e^{-n/2}t$.

Equations governing the saturation stage has relatively simple form when the trapped modes in the triplet are the same, or, in other words, when the equatorial wave with zonal wavenumber $k$ and frequency $\sigma$ interacts with the barotropic one having double frequency and zonal wavenumber, i.e. $k = 2k_1, \sigma = 2\sigma$. In this case the baroclinic amplitude obeys the Landau-type equation

$$\partial A / \partial T + LA_\psi A^* + M|A|^2 A = 0,$$

where the asterisk denotes the complex conjugate values and the constant coefficients $L, M$ depend on the wave parameters. If the amplitude $A$ is small then the cubic terms in (12) can be neglected and (12) is reduced to equation (9) with positive $K^{(+)\pm}$, i.e. at the initial stage the amplitude $A$ grows exponentially. With increasing time the cubic term becomes important, the growth slows down and the resulting amplitude tends to some constant limiting value since $Re M > 0$. Equation (12) has three stationary solutions given by the formulae $A_\pm = -LA_\psi / M$ and $A = 0$, the zero state being unstable and the solutions $A_\pm$ being stable and attractive. Thus a nonlinear saturation of a growing baroclinic wave always takes place in this case. Since $A_\pm$ are
of the order of unity, the saturated baroclinic amplitudes greatly exceed (by the factor $\varepsilon^{-1/2}$) the amplitude of the initial barotropic wave.

In general case when the trapped modes in the triad are different their evolution is governed by the system of two coupled Landau-type equation for the amplitudes $A_1, A_2$:

$$A_{1T} + \alpha_1 A_1^2 + \beta_1 |A_1|^2 A_1 + \gamma_1 |A_2|^2 A_1 = 0, \quad A_{2T} + \alpha_2 A_2^2 + \beta_2 |A_2|^2 A_2 + \gamma_2 |A_1|^2 A_2 = 0. \quad (13)$$

Here $\Re \beta_{m\parallel} \geq 0$ and at least one of the coefficients $\Re \gamma_m$ is non-negative which also provides saturation of the amplitudes. Generally, system (13) do not have stationary solutions, and the limiting saturated regime is harmonically oscillating, i.e. $A_1 = \overline{A}_1 e^{i\omega T}, \quad A_2 = \overline{A}_2 e^{-i\omega T}$ where $\overline{A}_{1,2}$ are complex constants and $\omega$ is a real frequency.

To study spatial modulation of the excited waves we introduce a hierarchy of slow spatial variables $X_1 = \varepsilon^{1/2} x, \quad X_2 = \epsilon x, ...$ into asymptotic representation (11) which takes the form

$$\psi = \psi_0(x, y, t, X_1, X_2, ..., T_1, T_2) + \psi_1(x, y, t, X_1, X_2, ..., T_1, T_2) + ... \quad , \quad (14a)$$

$$u, v, h = \varepsilon^{-1/2} (u_0, v_0, h_0)(x, y, t, X_1, X_2, ..., T_1, T_2) + (u_1, v_1, h_1)(x, y, t, X_1, X_2, ..., T_1, T_2) + ... \quad (14b)$$

Standard technique of eliminating secular terms gives in the case $k = 2\hat{k}, \quad \sigma = 2\hat{\sigma}$ the following equation for the baroclinic amplitude:

$$A_T - i\epsilon' \hat{k} A_{X,X_1} + L A_{\psi} A^* + M |A|^2 A = 0. \quad (15)$$

This equation is written in a reference frame moving with the group velocity of the baroclinic wave $c_{\psi} \hat{k}$ and is reduced to (12) when the modulation is absent. Equation (15) is an equation of the Ginzburg-Landau (GL) type and falls into the class of so-called resonantly forced GL equations describing various physical situations where parametric excitation of waves takes place. The above stationary solutions $A_{\pm}$ are also solutions to (15) and one can show that they are stable. The presence of two different stationary states suggests that solutions of the domain wall type exist, as for the similar GL equations (cf. Barashenkov, Woodford, 2005). Numerical simulations of (15) (Reznik, Zeitlin, 2007) corroborate this assumption.

In the general case of two baroclinic waves the spatially modulated amplitudes $A_1, A_2$ in the leading order obey the equations

$$A_{1T} + c_{\psi} (\hat{k}_1) A_{X,X_1} + \alpha_1 A_1^2 + \beta_1 |A_1|^2 A_1 + \gamma_1 |A_2|^2 A_1 = 0, \quad (16a)$$

$$A_{2T} + c_{\psi} (\hat{k}_2) A_{X,X_1} + \alpha_2 A_2^2 + \beta_2 |A_2|^2 A_2 + \gamma_2 |A_1|^2 A_2 = 0. \quad (16b)$$

Note that due to the presence of two different group velocities of the baroclinic waves, it is impossible to eliminate the terms with first spatial derivative, as it was done above in the case of a single baroclinic wave. The dispersive terms with second spatial derivatives appear as the next order corrections.

4 Interaction of barotropic Rossby wave with zonal equatorial current

Another type of interaction, which we study, is the interaction of the barotropic Rossby with baroclinic equatorial zonal flow. Let the lowest-order solution include barotropic wave (4) and the zonal flow

$$u, v, h = \varepsilon^{-\gamma} (\overline{u}_0, 0, \overline{h}_0)(y), \quad \overline{h}_0 y + y\overline{u}_0 = 0. \quad (17a, b)$$

Here the parameter $-1 < \gamma < 0$ measures the strength of the zonal flow. One can readily see from (2b,c), (3b,c) that the barotropic wave – mean flow interaction resonantly excites a trapped
equatorial mode with the zonal wavenumber and frequency coinciding with the corresponding parameters of barotropic wave. The amplitude of the baroclinic wave grows linearly in time according to equation (cf. (9))

\[ A_T = cA_y, T = \varepsilon^{\gamma+1} t, \]  

(18)

where the coefficients \( c \) and \( A_y \) are constant.

Mechanism of nonlinear saturation of this growth is similar to that in the triad interactions. The growing trapped mode gives rise to the growing secondary barotropic mode \( \psi_1 \) obeying (10), and then the interaction between the secondary barotropic mode with the zonal flow and trapped mode arrests the growth of \( A \) and results in saturation. The level of saturation depends on the strength of the initial zonal flow. The maximum level of saturation \( O(\varepsilon^{-1/2}) \) (like in the case of triad interactions) is reached for the zonal flow \( O(\varepsilon^{-1/2}) \). The solution in this case is represented in the asymptotic form (11), and the complete equation describing both the initial linear growth of the wave amplitude and its saturation is written as follows:

\[ A_T + aA + b|A|^2A = cA_y, \]  

(19)

where the complex coefficients \( a, b, c \) depend on the mean current profile and the trapped wave parameters. An important point is that \( \text{Re} a, \text{Re} b \geq 0 \); namely this fact provides the saturation of the trapped mode. Depending on the coefficients and the barotropic amplitude \( A_y \) equation (19) has either one or three stationary solutions. In the former case the stationary state is always stable and attracting, in the latter case two stationary states are stable and attracting and one is unstable.

To study the effects of spatial modulation of the incoming barotropic and excited trapped waves we again use asymptotic representation (14) assuming \( \gamma = 1/2 \). The “synthetic” modulation equations combining two leading orders for \( A \) and \( A_y \) follow:

\[ A_{T_1} + c_{gc} A_{X_1} + \sqrt{\varepsilon} \left[ -\frac{i}{2} c_{gc} A_{X_1, X_1} + aA + b|A|^2 A \right] = \sqrt{\varepsilon} cA_y, \]  

(20a)

\[ A_{y_{T_1}} + c_{gt} A_{yX_1} - \sqrt{\varepsilon} \frac{i}{2} c_{gr} A_{yX_1, X_1} = 0. \]  

(20b)

Here \( c_{gc}, c_{gt} \) are the group velocities of the baroclinic and barotropic waves, respectively, and the prime denotes differentiation with respect to the wave number \( k \).

A substantial difference between the trapped baroclinic Rossby and Yanai waves arises here. The group velocities \( c_{gc}, c_{gt} \) of the baroclinic and barotropic Rossby waves of the same frequency are practically the same, therefore in the case of Rossby waves excitation the equations (20) written in the reference frame moving with the common group velocity take the form

\[ A_{T_2} - \frac{i}{2} c_{gc} A_{X_1, X_1} + aA + b|A|^2 A = cA_y, \quad A_{y_{T_2}} - \frac{i}{2} c_{gr} A_{yX_1, X_1} = 0. \]  

(21)

At the same time, the group velocity of the Yanai wave may differ significantly from the group velocity of the barotropic wave of the same frequency. Therefore, in this case, the only situation where barotropic and baroclinic waves have possibility to interact significantly is that of “gentle” spatial modulation when the fields depend on \( X_2, T_2 \), and not on \( X_1, T_1 \). Then (20) are written as follows:

\[ A_{T_2} + c_{gc} A_{X_2} + aA + b|A|^2 A = cA_y, \quad A_{y_{T_2}} + c_{gt} A_{yX_2} = 0. \]  

(22)
The stationary solutions to (19) are also solutions to (20)-(22). If these equilibriums are multiple then numerical simulations of the systems (21), (22) reveal non-trivial spatio-temporal organization of the envelopes of the exciting waves (domain-wall defects, etc).

5 Conclusions

The physical picture arising from the presented results is as follows. In the linear approximation the equatorial waveguide is transparent for the barotropic Rossby waves. Due to nonlinear effects, the barotropic waves resonantly excite the baroclinic waveguide Rossby or Yanai modes with amplitudes growing (exponentially or linearly) in time. In its turn, the self-interaction of “trapped” part of solution gives rise to a growing correction $\varepsilon \psi_1$ to the initial barotropic wave $\psi_0$. The interaction between $\varepsilon \psi_1$ and the “trapped” part of solution arrests the growth of excited waveguide waves when $\varepsilon \psi_1$ becomes sufficiently large. The amplitudes of these waves are saturated but exhibit characteristic domain-wall structures like phase defects and “dark solitons”. The barotropic correction $\psi_1$ has the form of the reflected and transmitted wave structures spreading with time out of the equator i.e., a nonlinear scattering (absent in the linear approximation) of the barotropic wave takes place.

Thus the equator represents a semi-transparent waveguide where the waveguide modes may by resonantly excited by free external modes. This generation mechanism is very effective since the resulting amplitudes of the waveguide modes greatly exceed the amplitude of the free mode. We believe that the similar scenario is realized for other semi-transparent waveguides which are considered in geophysics and other branches of physics, e.g. in nonlinear optics or acoustics.

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References

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