Three-dimensionnal instabilities and transient growth of trailing vortices

Claire Donnadieu, Sabine Ortiz, Jean-Marc Chomaz & Paul Billant

CNRS - Ecole Polytechnique
LadHyX
Ecole Polytechnique, 91128 Palaiseau Cedex, FRANCE
claire.donnadieu@ladhyx.polytechnique.fr

Abstract:

An aircraft wake is made of counter-rotating vortices and is known to be affected by a long (Crow) and a short (elliptic) wavelength instabilities. Numerical investigations on the three-dimensionnal instabilities and transient growth of such dipole are performed. By means of a three-dimensionnal linear stability analysis, we retrieve the instability bands corresponding to the Crow and elliptic modes but we also observe less unstable oscillatory modes with very broad peaks. The transient growth of perturbations on this dipole, investigated by computing the optimal linear perturbations with a direct-adjoint technique, demonstrates the crucial role of the region of maximal strain at short time and of the hyperbolic point at intermediate time. Investigations on the three-dimensionnal dynamics of trailing vortices in stratified fluids are performed. The elliptic instability is almost unaffected by weak and moderate stratifications.

Résumé:

Le sillage d’un avion est constitué d’un paire de tourbillons contra-rotatifs et est affectée par une instabilité à grande (Crow) et à petite (elliptique) longueur d’onde. On réalise des études numériques sur les instabilités tridimensionnelles d’un tel dipole. Par une étude de stabilité linéaire tridimensionnelle, on retrouve les bandes d’instabilité correspondant aux modes de Crow et elliptiques mais on observe également des modes oscillants moins instables avec des pics très larges. Les croissances transitoires des perturbations sur ce dipole, qui sont étudiées en calculant les perturbations optimales linéaires par une technique direct-adjoint, démontrent le rôle crucial de la région où l’étirement est maximal aux temps courts et du point hyperbolique aux temps intermédiaires. Des études sur la dynamique tridimensionnelle des tourbillons de sillage d’avion en fluide stratifié sont réalisées. L’instabilité elliptique n’est pratiquement pas affectée par des stratifications faible et modérée.

Key-words:

instability ; optimal perturbations ; vortices

1 Introduction

Trailing vortices behind aircrafts consist of a horizontal pair of counter-rotating vortices propagating downwards. Depending on atmospheric conditions, such dipole can persist over a long time or be rapidly destroyed. If the vortex pair remains coherent, it can be hazardous to following aircrafts, especially during take-off and landing thus limiting the frequency between airplanes at airports. Studies of the dynamics of a pair of counter-rotating vortices in unstratified flows have shown that this vortex pair is unstable with respect to three-dimensional perturbations. Crow (1970) has discovered a long-wavelength instability, symmetric with respect to the plane separating the two vortices. The existence of a short-wavelength elliptic instability has been revealed by Tsai & Widnall (1976), Moore & Saffman (1975) and numerous articles ever since for both symmetric and antisymmetric modes. This instability, due to the elliptic deformation of the core of the vortices, is a resonant interaction between the strain and Kelvin waves of azimuthal wavenumbers $m = 1$ and $m = -1$ when both waves have the same frequency $\omega$.
and are particularly intense for $\omega = 0$.

However in many atmospheric situations, as such dipoles propagate downwards, they evolve under the influence of the stable stratification of the atmosphere and the three-dimensional dynamics of this vortex pair in stratified flow has yet received much less attention. Direct numerical simulations of Nomura et al. (2006) on the short-wavelength instability of a counter-rotating vortex pair in presence of stable stratification have suggested that the instability mechanism corresponds, despite the stratification, to the elliptic instability as in homogeneous media. The instability appears earlier than in the unstratified case, owing to the decrease due to the stratification of the separation distance between the vortices as they propagate downwards, decrease that induces larger ellipticity of the vortices and then enhances the instability.

In this paper, we perform a three-dimensionnal linear stability analysis of a Lamb-Oseen vortex pair in unstratified fluid in section 2. The transient growth of perturbations on this vortex pair, investigated by computing the optimal perturbations with the direct-adjoint technique introduced by Corbett & Bottaro (2000), is presented in section 3. In the case of weak stratification, the evolution of the two-dimensional flow due to stratification is slow and can be ignored in the leading order stability analysis and we perform a linear stability analysis of the frozen flow field at different instants in stratified fluid in section 4. In the case of strong stratification, the two-dimensional flow is unsteady and the optimal perturbations are computed at several times, with a direct-adjoint technique similar to the one used in the steady case and which takes into account the evolution of the flow. The results of this study are presented in section 5.

2 Linear three-dimensional instabilities in unstratified fluid

We investigate the three-dimensionnal instabilities of the pair of counter-rotating vortices represented on Figure 1, obtained by computing the two-dimensional evolution of initially two circular Lamb-Oseen vortices (i.e a gaussian distribution of vorticity), of circulation $\Gamma$, radius $a$ and with a separation distance $b$, as in Sipp et al. (1999). Since this base flow is symmetric with respect to the plane separating the two vortices, the linear stability modes may be decomposed in symmetric and antisymmetric parts. Figure 2 shows the real part of the growthrates $\sigma$ of the modes scaled by $2\pi b^2/\Gamma$ as function of the axial wavenumber $k_y$ scaled by the core radius $a$ for a dipole of aspect ratio $a/b = 0.206$ and for two Reynolds numbers based on the circulation of the vortices $Re_\Gamma = 10^5$ and $Re_\Gamma = 2000$. The first band of Figures 2(a) and 2(b) with a maximum at $k_y a = 0.19$ is the long-wavelength symmetric Crow instability. The three

Figure 1: Isovalues of (a) axial vorticity $\omega_{By}2\pi a^2/\Gamma$ and (b) the absolute value of the local strain rate $|\epsilon|$ of the base flow in the $(x,z)$ plane for $a/b = 0.206$. The stars represent the two hyperbolic points of the base flow and the arrowed lines correspond to the streamlines of the base flow.
Fig. 2: Scaled growthrates $\sigma^2 2\pi b^2 / \Gamma$ of symmetric (△) and antisymmetric (◦) modes as function of the scaled axial wavenumber $k_y a$ for (a) $\text{Re}_\Gamma = 10^5$ and (b) $\text{Re}_\Gamma = 2000$. Dashed line corresponds to the theory of Crow (1970) for the low wavenumber symmetric instability of a pair of vortex filaments. Continuous lines correspond to the inviscid theoretical prediction of Le Dizès & Laporte (2002) for a pair of Lamb-Oseen vortices in the limit $a/b = 0$.

Other peaks of Fig. 2(a) with maxima at $k_y a = 2.26$, $k_y a = 3.96$ and $k_y a = 5.64$ and the single peak of Fig. 2(b) with a maximum at $k_y a = 2.26$ corresponds to the elliptic instability. The growthrates of symmetric and antisymmetric modes are almost identical. The two broad lower peaks of Fig. 2(a) with maxima at $k_y a = 1.09$ and $k_y a = 4.2$ correspond to a novel oscillatory instability, which was not found by Sipp & Jacquin (2003). This instability exists for both symmetries and appears for sufficiently high Reynolds numbers. This oscillatory instability may be interpreted as an elliptic instability mode resulting from a resonance between the strain and Kelvin waves of azimuthal wavenumbers $m = 0$ and $|m| = 2$.

### 3 Optimal perturbations in unstratified fluid

We investigate the transient growth of perturbations on the vortex pair for the wavenumber corresponding to the maximum of the elliptic instability $k_y a = 2.26$ for $\text{Re}_\Gamma = 2000$ and for both symmetries. We use the technique introduced by Corbett & Bottaro (2000) to determine the optimal initial condition and the optimal response at finite time consisting of alternatively integrating forward in time the direct linearized Navier-Stokes (NS) operator and backward in time the adjoint NS operator. The Fig. 3 displays the enstrophy of the optimal perturbation and the optimal response at short time $t = 0.1$ (Fig. 3(a)) and intermediate time $t = 6$ (Fig. 3(b)) for the antisymmetric and the symmetric cases. At short time $t = 0.1$, the optimal perturbation is very similar the one the optimal response and, for both symmetries, the enstrophy is localized in the regions where the strain is maximum (dark red areas of Fig. 1(b)). At intermediate time $t = 6$, the spatial distributions of the optimal perturbation and the optimal response are different and concentrated respectively on the contracting and stretching manifold of one of the stagnation points of the base flow. The symmetric mode involves mainly the leading stagnation point (bottom star of Figs. 3(b)) and the symmetric mode the trailing stagnation point (top star of Figs. 3(b)).
Figure 3: Enstrophy of the optimal perturbation and optimal response in the \((x, z)\) plane for the anti-symmetric and the symmetric case at \(k_ya = 2.26\) and at times (a) \(t = 0.1\) and (b) \(t = 6\). White dots of figure (a) correspond to the points of maximum strain of the base flow. White arrowed lines of figure (b) correspond to the streamlines of the base flow and white stars represent the stagnation points of the base flow.

4 Linear stability analysis in stratified fluid

In a vertically stratified environment, the two-dimensional evolution of the horizontal vortex pair is affected by the stratification. As the dipole propagates downwards, opposite-sign vorticity appears around each vortex through baroclinic production and pushes the vortices towards one another, hence reducing the separation distance \(b\). In the case of weak stratification, i.e. for large Froude numbers \(Fr\), stratification acts on a long time scale \(1/N\) (\(N\) is the Brunt-Väisälä frequency) compared to the turnover time \(a/U\), we can consider that the basic state is quasi stationary and a linear stability analysis in stratified fluid describes the leading order evolution of the perturbation.

For short-wavelengths, we observe that elliptic instability modes persist. Figure 4 represents the non-dimensional growthrates \(\sigma 2\pi b^2/\Gamma\) as function of the non-dimensional axial wavenumber \(k_ya\), where \(a, b\) and \(\Gamma\) are the instantaneous values of the radius, the separation distance and the circulation of the vortices, for a dipole of initial aspect ratio \(a_0/b_0 = 0.2\), for three Froude numbers \(Fr = 10, Fr = 5\) and \(Fr = 2\), for \(Re_{\Gamma_0} = 2400\) at different instants such that once rescaled by \(N\), \(Nt = 1\) and \(Nt = 2\). We observe a band of unstable modes, which corresponds to the elliptic instability, with a maximum around \(k_ya = 2\) for all the cases. The growthrates rescaled by the instantaneous value of \(b^2/\Gamma\) are comparable for the three Froude numbers and at the two instants \(Nt = 1\) and \(Nt = 2\) meaning, since \(b^2/\Gamma\) decreases in time nearly as a function of \(Nt\), that the growthrate scaled by the initial value of \(b^2_0/\Gamma_0\) increases in time which corroborates the direct numerical simulations of Nomura et al. (2006).

5 Optimal perturbations in stratified fluid

In the case of strong stratification, i.e. for small Froude numbers, the unsteadiness of the flow makes the standard stability theory ineffective. In order to study the dynamics of this unsteady
flow, the optimal perturbations are determined at each time. The direct-adjoint technique developed for the unstratified case is still valid in the case of unsteady base flow and it has been adapted by taking into account this evolution and adding the density in the equations. Figure 5 represents the nondimensional growth rates $\sigma^2 \pi b^2 / \Gamma_0$ as function of the nondimensional axial wavenumber $k_y a$, where $a_0$, $b_0$ and $\Gamma_0$ are the initial values of the radius, the separation distance and the circulation of the vortices, for a dipole of initial aspect ratio $a_0/b_0 = 0.2$, for three Froude numbers $Fr = 10$, $Fr = 5$ and $Fr = 2$, for $Re_{\Gamma_0} = 2400$ at time $t = 4$. The radius $a$, the separation distance $b$ and the circulation $\Gamma$ of the vortices evolve as function of time, due to the presence of the stratification.

We observe broad instability bands for the symmetric and antisymmetric modes, indicating that the instability is less selective in $k_y$. The $Fr = 10$ and $Fr = 5$ bands have to be compared...
to the results of the linear stability analysis in stratified fluid at $Nt = 1$ and the $Fr = 2$ band to the ones at $Nt = 2$. For $Fr = 10$ and $Fr = 5$, the instability bands are similar with a maximum at $k_ya_0 = 1.6$ and comparable growthrates for both symmetries. The growthrates of symmetric and antisymmetric modes are close. The position of the maximum of the unstable band is in agreement with the theoretical prediction of the elliptic instability at $t = 4$ plotted on Figure 5 with a purple dashed line. For $Fr = 2$, the maximum is at a shorter axial wavenumber $k_ya_0 = 1.2$ and the growthrates are larger than for $Fr = 10$ and $Fr = 5$, which is in agreement with the results of the linear stability analysis of section 4. As the separation distance of the vortices decreases faster in time with increasing stratification, the vortices are closer at $t = 4$ for $Fr = 2$ than for $Fr = 10$ and $Fr = 5$. This induces a larger ellipticity of the vortices, hence enhancing the instability. Moreover, the growthrates of antisymmetric modes are larger than the symmetric ones for $Fr = 2$. This may be due to the increasing selection process of the antisymmetric mode with increasing $a/b$. Indeed, as the radius $a$ vary slightly and the separation distance $b$ is smaller at the same instant for stronger stratification, the aspect ratio of the dipole $a/b$ is larger for $Fr = 2$.

The instability mechanism is the elliptic instability, even for small Froude numbers for which the characteristic timescale is comparable to that of the instability. The elliptic theory predicts well the wavelength of the instability and, at the same instant, the growthrates of the instability are higher for stronger stratification since the elliptic deformation of the core of the vortices is enhanced due to the decrease of the separation distance between the vortices, as observed by Nomura et al. (2006).

References


Tsai, C.-Y., Widnall, S.E. 1976 The stability of short waves on a straight vortex filament in a weak externally imposed strain field J. Fluid Mech. 73 721-733