Influence of vibration on the onset of convection in two-layer system

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Abstract:
Current research is concentrated on the influence of translational vibrations on thermocapillary convection in two-layer system of viscous incompressible homogenous immiscible fluids, separated by deformable interface with surface tension coefficient dependent on temperature.
Linear stability was considered in the cases of single-frequency vibrations of finite and infinite frequencies, as well as multi-frequency oscillations. When frequency was finite, vibrations were considered vertical, and for large frequencies vibration direction was taken as arbitrary.
In the case of single-frequency oscillations continuous fractions method was applied, and for multi-frequency an eigenvalue problem for infinite matrix was solved. Parametric resonance regions, corresponding to synchronous and subharmonic disturbances, as well as oscillatory instability regions, were calculated. Neutral curves behaviour in amplitude-wave number and amplitude (minimized by wave number)-frequency axes was studied corresponding to different parameter combinations — layer densities, depths, viscosities, thermal conductivity coefficients, and surface tension. Interface elevation was calculated as well.
In the case of large frequencies averaging method was applied. It was shown that vibrations have stabilizing influence, with the strongest effect being when oscillations direction is vertical. Frequency values were calculated, which correspond to large frequency asymptotics.

Key-words:
Two-layer system; Marangoni convection; Vibration

1 Introduction

Researching parametric influence on any system leads to two most interesting cases — high-frequency oscillations and parametric resonance.
Influence of vertical high-frequency vibrations on the onset of convection in domain with solid boundary was first studied in Zenkovskaya, Simonenko (1966), where averaging method was applied to obtain a closed scleronomous system for averaged hydrodynamic field. In later works on vibrational convection the approach given in Zenkovskaya, Simonenko (1966) was used. In Zenkovskaya, Shleykel (2002) vibrational thermocapillary convection in horizontal fluid layer with deformable free boundary in the case of translational vibrations of arbitrary direction was studied. Problem, given in Zenkovskaya, Shleykel (2002), in the case of finite frequencies was discussed in Zenkovskaya et al. (2007). In this paper a method of continuous fractions, developed in V.I. Yudovich works and his followers Zenkovskaya, Yudovich (2004), Yudovich et al. (2004), was applied. Influence of vertical vibrations on the origin of waves on the interface in two-layer fluids was studied in Sekerzh-Zenkovich, Kalinichenko (1979), Sekerzh-Zenkovich (1983). Thermocapillary convection in two-layer systems was taken up in Birikh, Boushoueva (2001).
Current paper concentrates on investigation of vibrational thermocapillary convection in two-layer systems.
2 Problem statement

A system of two layers of immiscible viscous incompressible homogeneous fluids, bounded by two solid walls and separated with a deformable interface with surface tension coefficient \( \sigma = \sigma_0 - \sigma T \), dependent on temperature is considered. Coordinates will be denoted as \( x_1, x_2, x_3 \), \( x_3 \) axis is directed downwards and coincides with gravity vector. Point of origin is taken on undeformed interface. Upper wall is situated at depth \( x_3 = -H_2 \), lower — at depth \( x_3 = H_1 \). All values, corresponding to lower layer will be denoted with index 1, to upper layer — with index 2.

It is assumed that the system as a whole performs translational oscillations governed by the law \( x_3 = a f(\omega t) \) along vector \( \vec{s} = (\cos \varphi, 0, \sin \varphi) \), where \( f \) is 2\( \pi \)-periodic function with zero average, \( \varphi \) is vibration angle. Here \( \omega \) is the frequency, \( a = a(\omega) \) is the amplitude of vibrations.

Convection equations in dimensionless variables, written in cartesian coordinate system, rigidly fixed with oscillating system, have the form

\[
\rho_k \left( \frac{\partial \vec{v}^k}{\partial t} + (\vec{v}^k \cdot \nabla)\vec{v}^k \right) = -\nabla p^k + \mu_k \Delta \vec{v}^k + \rho_k \bar{g}(t), \quad \bar{g}(t) = Q_0 \bar{\gamma} - a_\omega^2 f''(\omega t) \bar{s}, \quad (1)
\]

\[
\text{div} \vec{v}^k = 0, \quad \frac{\partial T^k}{\partial t} + (\vec{v}^k \cdot \nabla)T^k = C_k \Delta T^k \quad (2)
\]

Here \( \vec{v}^k = (v^k_1, v^k_2, v^k_3) \) are relative velocities, \( p^k \) are the pressures, \( T^k \) are the temperatures, \( \gamma = (0, 0, 1) \).

Boundary conditions on the interface \( x_3 = \xi(x_1, x_2, t) \) are:

\[
\vec{v}^1 = \vec{v}^2, \quad \vec{v}^k \cdot \vec{n} = \frac{\partial \xi}{\partial t}, \quad \vec{\ell} = (-\xi_{x_1}, -\xi_{x_2}, 1), \quad \vec{n} = \frac{\vec{\ell}}{|\vec{\ell}|}, \quad (3)
\]

\[
-(p^1 - p^2)n_i + \left( \tau_{ij}^1 - \tau_{ij}^2 \right)n_j = -2K \sigma n_i - (\nabla \tau \sigma)_{ij}, \quad (4)
\]

\[
\nabla \Gamma (\sigma)_i = \frac{\partial \sigma}{\partial x_i} - \frac{\partial \sigma}{\partial x_k} n_k n_i, \quad 2K = \nabla_2 \frac{\nabla_2 \xi}{\sqrt{1 - |\nabla_2 \xi|^2}}, \quad \nabla_2 = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) \quad (5)
\]

\[
T^1 = T^2, \quad \kappa_1 \frac{\partial T^1}{\partial \vec{n}} - \kappa_2 \frac{\partial T^2}{\partial \vec{n}} + MT \text{div}_{\Gamma} \vec{v}^k = 0, \quad \sigma = C - MT, \quad \text{div}_{\Gamma} \equiv \nabla \Gamma \quad (6)
\]

On the solid outer boundaries of the system \( x_3 = h_1, -h_2 \):

\[
\vec{v}^k = 0, \quad B_{1k} \frac{\partial T^k}{\partial x_3} + B_{0k} T^k = b_k. \quad (7)
\]

Length, time, velocity, pressure and temperature scales are \( L, T, LT^{-1}, \rho L^2 T^{-2}, A \). \( \rho \) is the characteristic density scale, \( A \) is characteristic vertical temperature gradient. Specific scale selection will be made later.

The following dimensionless parameters are used: densities \( \hat{\rho}_k = \rho_k / \rho \), dynamic viscosities \( \hat{\mu}_k = \mu_k T / \rho L^2 \), depths, \( \hat{h}_k = H_k / L \), thermal diffusivity coefficients \( C_k = \chi_k T / L^2 \), thermal conductivity coefficients \( \hat{\kappa}_k = \kappa_k / \kappa \) (\( \kappa \) is the characteristic thermal conductivity coefficient scale), \( \hat{B}_{1k} = B_{1k} L \) is the dimensionless heat emission coefficient, \( \hat{b}_k = b_k A L \). \( Q_0 = g_0 T^2 / L \) is the constant part of gravity, \( \hat{\alpha} = a / \ell \) is the dimensionless vibration amplitude, \( \hat{\omega} = \omega T \) is the dimensionless vibration frequency, \( C = \sigma T^2 / \rho L^2 \) is the dimensionless surface tension coefficient, \( M = \sigma T A T^2 / \rho L^2 \) is the Marangoni number.
3  High frequency asymptotics

3.1  Krylov-Bogolubov averaging method

Further we consider the case when vibration frequency $\omega$ is large, and amplitude $a = b/\omega$ is small, so that velocity amplitude $b$ is finite. We also assume that the vibration period is smaller than characteristic hydrodynamic time, so that we can discard vibrational boundary layers.

Under such conditions the Krylov-Bogolubov averaging method can be applied to problem (1–7). A fast time $\tau = \omega t$ is introduced, and the solution is searched as a sum of slow and fast, having zero average in time, components:

$$
\tilde{v}^k = \tilde{v}^2(\tilde{x}, t) + b\omega^k(\tilde{x}, t) \tilde{f}(\tau), \quad p^k = \tilde{p}^k(\tilde{x}, t) + \omega b\rho_k \Phi^k(\tilde{x}, t) \tilde{f}''(\tau),
$$

$$
T^k = \tilde{T}^2(\tilde{x}, t) - \frac{1}{\omega} b(\tilde{w}^k(\tilde{x}, t), \nabla \tilde{T}^k(\tilde{x}, t)) f(\tau),
$$

$$
\xi = \tilde{\xi}(x_1, x_2, t) + \frac{1}{\omega} b(\tilde{w}^k(\tilde{x}, t), \tilde{\ell}(x_1, x_2, t)) f(\tau)
$$

where amplitudes $\tilde{w}^k$ and $\Phi^k$ are governed by equations:

$$
\tilde{w}^k = -\nabla \Phi^k + \tilde{s}, \quad \text{div} \tilde{w}^k = 0
$$

As a result of averaging method application, we obtain closed system for slow variables (bar is omitted):

$$
\frac{\partial \bar{\tilde{v}}^k}{\partial \tau} + (\tilde{v}^k, \nabla) \tilde{v}^k = -\frac{1}{\rho_k} \nabla p^k + \nu_k \Delta \tilde{v}^k + Q_0 \gamma + \text{Re}^2(\tilde{w}^k, \nabla) \nabla \Phi^k, \quad (8)
$$

$$
\text{div} \bar{\tilde{v}}^k = 0, \quad \frac{\partial T^k}{\partial \tau} + (\tilde{v}^k, \nabla) T^k = C_k \Delta T^k, \quad \bar{\tilde{w}}^k - \tilde{s} = -\nabla \bar{\Phi}^k, \quad \text{div} \bar{\tilde{w}}^k = 0. \quad (9)
$$

Here $\text{Re}^2 = b^2 \langle f'^2 \rangle, \langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\tau) d\tau$.

On solid walls $x_3 = h_1, -h_2$:

$$
\bar{\tilde{v}}^k = 0, \quad w^k_3 = 0, \quad B_{1k} \frac{\partial T^k}{\partial x_3} + B_{0k} T^k = b_k \quad (10)
$$

On averaged interface $x_3 = \xi(x_1, x_2, t)$:

$$
\tilde{v}^1 = \tilde{v}^2, \quad w^1_n = w^2_n, \quad (\tilde{v}^k, \cdot) = \frac{\partial \xi}{\partial \nu},
$$

$$
(\tau^1_{ij} - \tau^2_{ij}) n_j - (p_1 - p_2 - \tau_v) n_i = -2K_\sigma n_i - (\nabla I \sigma)_i \quad (11)
$$

$$
\tau_v = \text{Re}^2(\rho_1 \frac{\partial \Phi^1}{\partial x_3} \tilde{w}^1 - \rho_2 \frac{\partial \Phi^2}{\partial x_3} \tilde{w}^2, \tilde{v}),
$$

$$
T^1 = T^2, \quad \kappa_1 \frac{\partial T^1}{\partial \nu} - \kappa_2 \frac{\partial T^2}{\partial \nu} + MT^k \text{div} I \tilde{v}^k = 0, \quad \rho_1 \tilde{\Phi}^1 = \rho_2 \tilde{\Phi}^2. \quad (12)
$$

3.2  Equilibrium solution. Stability problem

Averaged system (8–14) has an equilibrium solution:

$$
\bar{\tilde{v}}^{0k} = 0, \quad T^{0k} = A_k z, \quad \xi^0 = 0, \quad \bar{w}^0 = (\cos \varphi, 0, 0), \quad \Phi^{0k} = z \sin \varphi
$$
After applying linearization technique, setting \( \tilde{v}^k = \tilde{v}^{0k} + \tilde{v}^k, \xi = \xi^0 + \eta, p^k = P^k + p^{0k}, T^k = \theta^k + T^{0k}, w^k = w^{0k} + \tilde{W}^k, \Phi^k = \Phi^{0k} + \tilde{\Phi}^k \) (tilde is omitted further), introducing flow functions \( u^k_1 = \frac{\partial \tilde{v}^k}{\partial x}, u^k_2 = -\frac{\partial \tilde{w}^k}{\partial x}, \tilde{W}_1^k = \frac{\partial \tilde{v}^k}{\partial x}, \tilde{W}_3^k = -\frac{\partial \tilde{w}^k}{\partial x} \) (bar is omitted further), eliminating pressures \( P^k \) and functions \( \Phi^k \), separating variable \( x \) by substitution \( \psi^k(x, z, t), \omega^k(x, z, t), \theta^k(x, z, t), \eta(x, t)) = e^{i\alpha x}e^{is\alpha}(i\alpha \psi^k(z), i\alpha \omega^k(z), \theta^k(z), \eta(t)) \) and obtaining explicit expression for \( \omega^k(z) \) the resulting spectral problem can be written as:

\[ L \psi^k = \nu_k L^2 \psi^k, \quad \lambda \theta^k + \alpha^2 A_k \psi^k = C_k L \theta^k \]

On \( x_3 = h_1, -h_2 \):

\[ \psi = 0, \quad D \psi = 0, \quad B_{1k} \theta^k + B_{0k} \theta^k = 0 \]

On linearized interface \( x_3 = 0 \):

\[ \hat{\psi}^1 = \psi^2, \quad D \hat{\psi}^1 = D \psi^2, \quad \alpha^2 \hat{\psi}^k = \lambda \eta, \]

\[ \mu_1 D^2 \psi^1 - \mu_2 D^2 \psi^2 + \alpha^2 (\mu_1 - \mu_2) \psi^1 = M(\theta^k + A_k \eta), \]

\[ 3 \alpha^2 (\mu_1 - \mu_2) D^2 \psi^1 + \rho_1 \lambda D \psi^1_1 - \rho_2 \lambda D \psi^1_2 - (\mu_1 D^3 \psi^1 - \mu_2 D^3 \psi^2) - Q_0 (\rho_1 - \rho_2) \eta - (C \alpha^2 + R \alpha \alpha) \eta = 0, \]

\[ \theta^1 + A_1 \eta = \theta^2 + A_2 \eta, \]

\[ \kappa_1 D \theta^1 = \kappa_2 D \theta^2. \]

Here \( R = R e^2 \sin^2 \varphi \frac{(\rho_1 - \rho_2)^2}{\rho_1 \sinh h_1 + \rho_2 \sinh h_2} \) is the vibration-generated surface tension.

This problem can be numerically solved for neutral curves in \((M, \alpha)\) plane for different parameter values.

4 Finite frequencies, vertical oscillations

For the finite frequencies case we take that vibrations are vertical, that is, \( \varphi = \pi/2 \).

4.1 Quasiequilibrium solution. Stability problem

The problem (1–7) has quasiequilibrium solution:

\[ \tilde{v}^{0k} = 0, \quad \xi^0 = 0, \quad p^{0k} = \rho_k g(t) z + \phi(t), \quad T^{0k} = A_k z \]

where \( \phi(t) \) is an arbitrary function.

After applying linearization technique, setting \( \tilde{v}^k = \tilde{v}^{0k} + \tilde{v}^k, \xi = \xi^0 + \eta, p^k = P^k + p^{0k}, T^k = \theta^k + T^{0k} \), introducing flow functions \( u^k_1 = \frac{\partial \tilde{v}^k}{\partial x}, u^k_2 = -\frac{\partial \tilde{w}^k}{\partial x}, \tilde{W}_1^k = \frac{\partial \tilde{v}^k}{\partial x}, \tilde{W}_3^k = -\frac{\partial \tilde{w}^k}{\partial x} \), eliminating pressures \( P^k \) separating variable \( x \) by substitution \( \psi^k(x, z, t), \theta^k(x, z, t), \eta(x, t)) = e^{i\alpha x}e^{is\alpha}(i\alpha \psi^k(z), i\alpha \theta^k(z), \theta^k(z), \eta(t)) \) the resulting spectral problem can be written as:

\[ \rho_k \frac{\partial}{\partial t} (D^2 - \alpha^2) \psi^k = \mu_k (D^2 - \alpha^2)^2 \psi^k, \quad \frac{\partial \theta^k}{\partial t} = C_k (D^2 - \alpha^2) \theta^k - A_k \alpha^2 \psi^k \quad (15) \]

On \( z = h_1, -h_2 \):

\[ \psi^k = 0, \quad D \psi^k = 0, \quad B_{1k} \theta^k + B_{0k} \theta^k = 0. \quad (16) \]
On linearized interface $z = 0$:
\[
\psi^1 = \psi^2, \quad D\psi^1 = D\psi^2, \quad \alpha^2\psi^k = \eta^k, \quad \theta^1 + A_1\eta = \theta^2 + A_2\eta, \quad (17)
\]
\[
\kappa_1D\theta^1 = \kappa_2D\theta^2, \quad \mu_1D^2\psi^1 - \mu_2D^2\psi^2 + \alpha^2(\mu_1 - \mu_2)\psi^1 = Ma(\theta^k + A_k\eta), \quad (18)
\]
\[
3\alpha^2(\mu_1 - \mu_2)D\psi^1 + \rho_1D\psi^1 - \rho_2D\psi^2 - (\mu_1D^3\psi^1 - \mu_2D^3\psi^2) - 
- Q(t)(\rho_1 - \rho_2)\eta - C\alpha^2\eta = 0, \quad Q(t) = Q_0 - a\omega^2f''(\omega t) \quad (19)
\]

### 4.2 Floquet solutions

Further we shall consider the case, when system performs oscillations having Fourier series expansion $f(\omega t) = \sum_{j=-\infty}^{+\infty} f_j e^{ij\omega t}$. Separating time, we shall seek the unknowns in the form of infinite Fourier series:
\[
(\psi^k(z, t), \theta^k(z, t), \eta(t)) = e^{\lambda t} \sum_{n=\infty}^{+\infty} (\psi^k_n(z), \theta^k_n(z), c_n) e^{in\omega t} \quad (20)
\]

Here $\lambda = \lambda_c + i\lambda_i$ is a complex number — Floquet multiplier.

Substituting (20) into the problem (15–19) an infinite system of ordinary differential equations is obtained:
\[
\rho_k\lambda_n(D^2 - \alpha^2)\psi^k_n = \mu_k(D^2 - \alpha^2)\psi^k_n, \quad \lambda_n\theta^k_n = C_k(D^2 - \alpha^2)\theta^k_n - A_k\alpha^2\psi^k_n
\]

On $z = h_1, -h_2$:
\[
\psi^k_n = 0, \quad D\psi^k_n = 0, \quad B_{1k}D\theta^k_n + B_{0k}\theta^k_n = 0.
\]

On linearized interface $z = 0$:
\[
\psi^1_n = \psi^2_n, \quad D\psi^1_n = D\psi^2_n, \quad \alpha^2\psi^k_n = \lambda_n c_n, \quad \theta^1_n + A_1\eta = \theta^2_n + A_2c_n,
\]
\[
\kappa_1D\theta^1_n = \kappa_2D\theta^2_n, \quad \mu_1D^2\psi^1_n - \mu_2D^2\psi^2_n + \alpha^2(\mu_1 - \mu_2)\psi^1_n = Ma(\theta^k_n + A_kc_n),
\]
\[
3\alpha^2(\mu_1 - \mu_2)D\psi^1_n + (\rho_1 - \rho_2)\lambda_nD\psi^1_n - (\mu_1D^3\psi^1_n - \mu_2D^3\psi^2_n) - 
- ((\rho_1 - \rho_2)Q_0 + C\alpha^2)c_n - 2a\omega^2(\rho_1 - \rho_2) \sum_{j+k=n} f_j c_k = 0
\]

For each $n$ for unknowns $c_n$ we obtain an expression:
\[
M_n(\lambda)c_n = -2q \sum_{j+k=n} f_j c_k, \quad n = 0, \pm 1, \pm 2, \ldots \quad q = (\rho_1 - \rho_2)a\omega^2\alpha \quad (21)
\]

where $M_n(\lambda)$ is not given due to its unhandedness.

In general this problem can be solved as an eigenvalue problem for $q$. When $f(\omega t)$ is a harmonic function, for example, $f = \cos \omega t$ (therefore its Fourier decomposition coefficients are: $c_{-1} = c_1 = 1/2, c_n = 0, n = -\infty.. + \infty, n \neq -1, 1$) the system (21) becomes three-diagonal. For such system it is possible to write a dispersion relation for $\lambda$ in explicit form using continuous fractions approach, presented in Zenkovskaya, Yudovich (2004). Dispersion relation, obtained from (21), has the form (here and further $f(\omega t) = \cos \omega t)$:
\[
-M_n + \frac{-q^2}{M_{n+1} + \frac{-q^2}{M_{n+2} + \ldots}} = \frac{-q^2}{M_{n-1} + \frac{-q^2}{M_{n-2} + \ldots}} \quad (22)
\]
When $\lambda = 0$ the equation (22) is:

$$\text{Re} \left( \frac{-q^2}{M_1 + \frac{-q^2}{M_2 + \ldots}} \right) = -\frac{M_0}{2}$$

If $\lambda = i\omega/2$ it leads to:

$$\left| \frac{-q^2}{M_1 + \frac{-q^2}{M_2 + \ldots}} \right|^2 = q^2$$

Following Zenkovskaya et al. (2007), neutral curves were computed for all three main cases of stability loss: synchronous, subharmonic and quasiperiodic.

5 Conclusions

The purpose of this paper is not only obtaining of qualitative results on vibrational influence, but also demonstration of effective methods of approaching such problems — averaging method and continuous fractions method. They allow to minimize the amount of calculations.

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