Transcritical Flow Over a Step

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Abstract:

In both the ocean and the atmosphere, the interaction of a density stratified flow with topography can generate large-amplitude, horizontally propagating internal solitary waves. Often these waves appear as a wave-train, or undular bore. In this talk we focus on the situation when the flow is critical, that is, the flow speed is close to that of a linear long wave mode. In the weakly nonlinear regime, this is modeled by the forced Korteweg de Vries equation. We will review how Whitham’s modulation theory has been applied to obtain an analytical description of undular bores for flow over isolated obstacles, and then show how the same approach can be used for flow over a step.

Résumé:

L’interaction d’un écoulement stratifié en densité avec la topographie peut générer des ondes solitaires de grande amplitude se propageant horizontalement dans l’atmosphère ou l’océan. Souvent, ces ondes apparaissent comme un train d’ondes, ou un mascaret. Dans cette présentation, nous focalisons sur la situation où l’écoulement est critique, c’est-à-dire, la vitesse du courant est proche de celle d’un mode relatif à une onde linéaire de grande longueur d’onde. Dans le régime faiblement nonlinéaire, celui-ci est modélisé par l’équation de Korteweg-de-Vries forcée. Nous passerons en revue comment la théorie de la modulation de Whitham fut appliquée afin d’obtenir une description analytique des mascaret pour un écoulement au-dessus d’obstacles isolés, et ensuite, nous montrerons comment la même approche peut être utilisée pour un écoulement au-dessus d’une marche.

Key-words:

Solitary waves; undular bore; flow over topography.

1 Introduction

Solitary waves are nonlinear waves of quasi-permanent form, first observed by Russell (1844). Later Boussinesq (1871) and Rayleigh (1876) established a theoretical model, and then Korteweg & de Vries (1895) derived the well-known equation which now bears their names. But it was not until the second half of the twentieth century that it was realised that the Korteweg-de Vries equation was a valid model for solitary waves in a wide variety of physical contexts. Of principal concern here are the large-amplitude solitary waves which propagate in density-stratified fluids such as the ocean and atmosphere (see, e.g., Grimshaw (2001) Holloway, Pelinovsky & Talipova (2001) and Rottman & Grimshaw (2001)). They owe their existence to a balance between nonlinear wave-steepening effects and linear wave dispersion, and hence can be effectively modeled by nonlinear evolution equations of the Korteweg-de Vries (KdV) type.

Often, these waves are generated by critical flow over topography, and in this situation the waves appear as upstream and downstream wavetrains, each having the character of an undular bore. In this situation the appropriate model equation is the forced Korteweg-de Vries equation (fKdV), given by (see Akylas (1984) for water waves and Grimshaw & Smyth (1986)
for internal waves)

\[-\frac{1}{c}(A_t + \Delta A_x) + \mu AA_x + \gamma A_{xxx} + \frac{1}{2}F_x = 0,\]

(1)

Here \(A(x, t)\) is the amplitude of the wave, and \(x, t\) are space and time variables respectively. The coefficients \(c\) and \(\Delta\) are the relevant linear long wave speed (equal to the flow speed \(U\) at criticality) and the departure from criticality (i.e. \(\Delta = U - c\)); the coefficients \(\mu\) and \(\gamma\) of the nonlinear and dispersive terms are determined by the waveguide properties of the specific physical system being considered, while the forcing term \(F(x)\) is the projection of the topography onto the relevant waveguide mode.

In this paper, we shall review briefly the theory of the undular bore based on the Korteweg-de Vries equation in Section 2, and then in Section 3 its application to the description of undular bores generated by flow over an isolated obstacle. Then in Section 4 we extend that theory to describe how upstream undular bores are generated by flow over a forward-facing topographic step, and downstream by a backward-facing step.

2 Undular bore

An undular bore is an oscillatory transition between two different basic states. Here, we are concerned with non-dissipative flows, in which case an undular bore is intrinsically unsteady. A simple representation can be obtained from the solution of the unforced KdV equation, that is (1) with \(F(x) \equiv 0\), written here in the canonical form,

\[A_t + 6AA_x + A_{xxx} = 0.\]

(2)

with the initial condition that \(A = A_0 \, H(-x)\) (\(H(x)\) is the Heaviside function), where we assume at first that \(A_0 > 0\). Although the solution can in principle be obtained through the inverse scattering transform, it is more instructive to use the asymptotic method developed by Gurevich & Pitvaeskii (1974) and Whitham (1974). In this approach, the solution of (2) is represented as the modulated periodic wave train

\[A = a \{b(m) + cn^2(\gamma(x-Vt); m)\} + d,\]

(3)

where \(b = \frac{1 - m}{m} - \frac{E(m)}{mK(m)}\), \(a = 2m\gamma^2\), \(V = 6d + 2a \left\{ \frac{2 - m}{m} - \frac{3E(m)}{mK(m)} \right\} \). \(4\)

Here \(cn(x;m)\) is the Jacobian elliptic function of modulus \(m, 0 < m < 1\), \(K(m), E(m)\) are the elliptic integrals of the first and second, \(\gamma\) is a wavenumber such that the spatial period is \(2K(m)/\gamma\) and \(a, d\) are the amplitude and mean level respectively. As the modulus \(m \to 1\), this becomes a solitary wave, since then \(b \to 0\) and \(cn^2(x) \to \text{sech}^2(x)\), but as \(m \to 0\) it reduces to sinusoidal waves of small amplitude \(a \sim m\) and wavenumber \(2\gamma\).

The asymptotic method of Gurevich & Pitvaeskii (1974) and Whitham (1974) is to let the expression (3) describe a modulated periodic wavetrain in which the amplitude \(a\), the mean level \(d\), the speed \(V\) and the wavenumber \(\gamma\) are all slowly varying functions of \(x\) and \(t\). The relevant asymptotic solution corresponding to the step-discontinuity initial condition can then be constructed in terms of the similarity variable \(x/t\), and is given by

\[
\frac{x}{t} = 2A_0 \left\{ 1 + m - \frac{2m(1 - m)(K(m))}{E(m) - (1 - m)K(m)} \right\}, \quad \text{for} \quad -6A_0 < \frac{x}{t} < 4A_0, \]

(5)

\[
a = 2A_0 m, \quad d = A_0 \left\{ m - 1 + \frac{2E(m)}{K(m)} \right\}. \]

(6)
Ahead of the wavetrain where $x/t > 4A_0$, $A = 0$ and at this end, $m \to 1$, $a \to 2A_0$ and $d \to 0$; the leading wave is a solitary wave of amplitude $2A_0$ relative to a mean level of 0. Behind the wavetrain where $x/t < -6A_0$, $A = A_0$ and at this end $m \to 0$, $a \to 0$, and $d \to A_0$; the wavetrain is now sinusoidal with a wavenumber $2\gamma = 2\sqrt{A_0}$. Further, it can be shown that on any individual crest in the wavetrain, $m \to 1$ as $t \to \infty$. In this sense, the undular bore evolves into a train of solitary waves.

If $A_0 < 0$ in the initial condition then an “undular bore” solution analogous to that described by (3, 5) does not exist. Instead, the asymptotic solution is a rarefraction wave,

$$A = 0 \text{ for } x > 0, \ A = \frac{x}{6t} \text{ for } A_0 < \frac{x}{6t} < 0, \ A = A_0, \text{ for } \frac{x}{6t} < A_0(< 0).$$

Small oscillatory wavetrains are needed to smooth out the discontinuities in $A_x$ at $x = 0$ and $x = -6A_0$ (for further details, see Gurevich & Pitvaeskii (1974)).

3 The generation of undular bores by flow over localized topography

The fKdV equation (1) can be put into the canonical form

$$-A_t - \Delta A_x + 6AA_x + A_{xxx} + F_x(x) = 0.$$  

(8)

This is to be solved with the initial condition that $A(x, 0) = 0$, which corresponds to a slow introduction of the topographic obstacle. The following summary is based on Grimshaw & Smyth (1986).

First we consider the typical solution of (8) when the forcing $F(x)$ is positive and localized, with a maximum of $F_M > 0$. The solution is characterised by upstream and downstream wavetrains connected by a locally steady solution over the obstacle. For supercritical flow ($\Delta < 0$) the upstream wavetrain weakens, and for sufficiently large $|\Delta|$ detaches from the obstacle, while the downstream wavetrain intensifies and for sufficiently large $|\Delta|$ forms a stationary lee wave field. On the other hand, for supercritical flow ($\Delta > 0$) the upstream wavetrain develops into well-separated solitary waves while the downstream wavetrain weakens and moves further downstream (for more details see Grimshaw & Smyth (1986) and Smyth (1987)).

The origin of the upstream and downstream wavetrains can be found in the structure of the locally steady solution over the obstacle. In the transcritical regime this is characterised by a transition from a constant state $A_- > 0$ upstream of the obstacle to a constant state $A_+ < 0$ downstream of the obstacle. It is readily shown that $\Delta = 3(A_+ + A_-)$ independently of the details of the forcing term $F(x)$. Explicit determination of $A_+$ and $A_-$ requires some knowledge of the forcing term $F(x)$. However, in the “hydraulic” limit when the linear dispersive term in (8) can be neglected, it is readily shown that in the transcritical regime $6A_\pm = \Delta \mp (12F_M)^{1/2}$ where $|\Delta| < (12F_M)^{1/2}$. Thus upstream of the obstacle there is a transition from 0 to $A_-$, while downstream the transition is from $A_+$ to 0.

Both transitions are resolved by “undular bore” solutions as described in section 2. That in $x < 0$ is exactly described by (3) to (6) with $x$ replaced by $\Delta t - x$, and $A_0$ by $A_-$. It occupies the zone $\Delta - 4A_- < x/t < \min\{0, \Delta + 6A_\}$. Note that this upstream undular bore is constrained to lie in $x < 0$, and hence is only fully realised if $\Delta < -6A_-$, thus defining the regime $-(12F_M)^{1/2} < \Delta < -\frac{1}{3}(12F_M)^{1/2}$ where a fully developed undular bore solution can develop upstream. On the other hand, the regime $\Delta > -6A_-$ or $-\frac{1}{5}(12F_M)^{1/2} < \Delta < (12F_M)^{1/2}$ is where the upstream undular bore is only partially formed, and is attached to the obstacle. In this case the modulus $m$ of the Jacobian elliptic function varies from 1 at the leading edge (thus describing solitary waves) to a value $m_-$ (< 1) at the obstacle, where $m_-$ can be found from (5)
by replacing \(x/t\) with \(\Delta\) and \(A_0\) with \(A_-\). For instance, when \(\Delta = 0, m_- = 0.8\) independently of \(F_M\).

The transition in \(x > 0\) can also be described by (3) to (6) where we now replace \(x\) with \((\Delta - 6A_+) t - x, A_0\) with \(-A_+\), and \(d\) with \(-d - A_+\). This “undular bore” solution now occupies the zone \(\max\{0, \Delta - 2A_+\} < x/t < \Delta - 12A_+\). Here, this downstream wavetrain is constrained to lie in \(x > 0\), and hence is only fully realised if \(\Delta > 2A_+\), which defines exactly the same regime as that for the case when the upstream undular bore is attached to the obstacle. On the other hand, in the regime where the upstream undular bore is detached from the obstacle, the downstream undular bore is attached to the obstacle, and the modulus \(m\) varies from \(m_+ (< 1)\) at the obstacle, where \(m_+\) can be found from (5) by replacing \(x/t\) with \(\Delta - 6A_+\) and \(A_0\) with \(-A_+\) to \(m = 0\) at the trailing edge. Further, a stationary lee wavetrain may develop just behind the obstacle (for further details, see Smyth, 1987). At the transition to non-resonant subcritical flow, where \(\Delta = -\sqrt{12F_M}, m_+ = 0.96\) independently of \(F_M\).

For the case when the obstacle has negative polarity (that is \(F(x)\) is negative, and non-zero only in the vicinity of \(x = 0\)), the upstream and downstream solutions are qualitatively similar. However, the solution in the vicinity of the obstacle remains transient, and this causes a modulation of the “undular bore” solutions.

### 4 Generation of solitary waves by flow over a step

Here we consider the situation where the forcing term in (1) has a step-like structure, that is, \(F(x) = 0\) for \(x < 0\) and \(F(x) = F_M\) for \(x > L\), while \(F(x)\) varies monotonically in \(0 < x < L\). A positive (negative) step has \(F_M > 0 (< 0)\). Strictly \(F(x)\) should return to zero for some \(L_1 \gg L\). Here we ignore this, and in effect assume that \(L_1 \to \infty\). In practice it means that the solutions constructed below are only valid for some limited time, determined by how long it takes for a disturbance to travel a distance \(L_1\).

We shall sketch how the solution for the localized forcing described above becomes modified for a step, and adapt the approach used by Grimshaw & Smyth (1986), where we first construct the local steady-state solution in the forcing region, \(0 < x < L\), using the “hydraulic” limit. In this limit \(A = A(x), 0 < x < L\) while \(A = A_-\) for \(x < 0\) and \(A = A_+\) for \(x > L\). It is readily found that \(-\Delta A + 3A^2 + F = C\). Here the constant \(C\) is determined by considering the long-time limit of the unsteady hydraulic solution, as in Grimshaw & Smyth (1986). But note that \(C = -\Delta A_- + 3A_-^2 = -\Delta A_+ + 3A_+^2 + F_M\), giving a connection between \(A_-\) and \(A_+\).

Suppose first that the step is positive, \(F_M > 0\). Then the local hydraulic solution is,

\[
\begin{align*}
\Delta \leq 0 : \quad 6A_- &= \Delta + (\Delta^2 + 12F_M)^{1/2}, \quad 6A_+ = 0, \\
0 < \Delta < (12F_M)^{1/2} : \quad 6A_- &= \Delta + (12F_M)^{1/2}, \quad 6A_+ = \Delta, \\
\Delta > (12F_M)^{1/2} : \quad 6A_- = 0 \quad 6A_+ = \Delta - (\Delta^2 - 12F_M)^{1/2}.
\end{align*}
\]

Here the constant \(C = F_M, F_M - \Delta^2/12, 0\) respectively. In all cases, the upstream solution \(A_- > 0\) is a “shock” in the hydraulic limit (although in (11) the shock has zero strength and so can be ignored), which needs to be replaced with an “undular bore” as in section 3. But importantly note that the upstream elevation \(A_-\) is different from that found for flow over a localized obstacle, except at exact resonance when \(\Delta = 0\). The undular bore is again given by (3) to (6) with \(x\) replaced by \(\Delta t - x\) and \(A_0\) by \(A_-\) and occupies the zone \(\Delta - 4A_- < x/t < \min\{0, \Delta + 6A_-\}\). For a fully detached undular bore, \(\Delta + 6A_- < 0\), and combining this criterion with (9, 10), we get the regime \(\Delta < -2(F_M)^{1/2} < 0\). On the other hand the regime where \(\Delta + 6A_- > 0\) but \(\Delta - 4A_- < 0\), or \(-2(F_M)^{1/2} < \Delta < (12F_M)^{1/2}\) is where the upstream undular bore is only partially formed and is attached to the obstacle. As for the localized forcing
case described in section 3, the modulus \( m \) of the Jacobian elliptic function varies from 1 at the leading edge (thus describing solitary waves) to a value \( m_- \) (< 1) at the obstacle, where \( m_- \) can be found from (5) by replacing \( x/t \) with \( \Delta \) and \( A_0 \) with \( A_- \). As before, when \( \Delta = 0 \), \( m_- = 0.8 \) independently of \( F_M \). Downstream, for \( \Delta > 0 \), \( A_+ > 0 \), and the hydraulic solution is terminated by a rarefraction wave; hence, no undular bore solution is needed. Instead a weak oscillatory wave train is needed to smooth the corners. When \( \Delta < 0 \), \( A_+ = 0 \), and no undular bore or rarefraction wave is needed.

Next consider a negative step, \( F_M < 0 \). Now the local hydraulic solution is,

\[
\Delta \geq 0 : \quad 6A_+ = \Delta - (\Delta^2 - 12F_M)^{1/2}, \quad 6A_- = 0, \tag{12}
\]
\[
-(-12F_M)^{1/2} < \Delta < 0 : \quad 6A_+ = \Delta - (-12F_M)^{1/2}, \quad 6A_- = \Delta, \tag{13}
\]
\[
\Delta < -(-12F_M)^{1/2} : \quad 6A_+ = 0, \quad 6A_- = \Delta + (\Delta^2 + 12F_M)^{1/2}. \tag{14}
\]

Here the constant \( C = 0, -\Delta^2/12, F_M \) respectively. In all cases the downstream solution \( A_+ < 0 \) is a shock (in (14) the shock has zero strength), and needs to be replaced by an undular bore solution as in Section 3, but again note that the downstream depression \( A_+ \) differs from that found for flow over a localized obstacle, except for \( \Delta = 0 \). The undular bore is given by (3) to (6) with \( x \) replaced by \( (\Delta - 6A_+)t - (x - L) \), \( A_0 \) by \(-A_+\) and \( d \) with \( d - A_+ \). It occupies the zone \( \max \{0, \Delta - 2A_+\} < (x - L)/t < \Delta - 12A_+ \). For a fully detached undular bore, \( \Delta - 2A_+ > 0 \), and combining with the criteria (13, 14) we get the regime \( \Delta > -(-3F_M)^{1/2} \). On the other hand, the regime where \( \Delta - 2A_+ < 0 \) but \( \Delta - 12A_+ > 0 \), or \(-(-12F_M)^{1/2} < \Delta < -(-3F_M)^{1/2} < 0 \), is where the undular bore is only partially formed and is attached to the step in the same manner described above in Section 3. That is, the modulus \( m \) varies from \( m_+ (< 1) \) at the step, where \( m_+ \) can be found from (5) by replacing \( x \) with \( \Delta - 6A_+ \) and \( A_0 \) with \(-A_+\) to \( m = 0 \) at the trailing edge. Further, a stationary lee wavetrain may develop just behind the obstacle. At the transition to non-resonant subcritical flow, where \( \Delta = -\sqrt{-12F_M} \),
again \( m_+ = 0.96 \) independently of \( F_M \). For \( \Delta < 0, A_- < 0 \) and the hydraulic solution is terminated by a rarefaction wave, and hence no undular bore is needed (but an oscillatory wave train is needed to smooth out the corners. For \( \Delta > 0 \) the upstream solution is zero.

These predictions are confirmed by direct numerical simulations of the fKdV equation (8) for the case when the forcing term is a step with \( F_M > 0 \), terminated by an analogous step down at the point \( x = L_1 >> L \). A typical scenario is shown in Figure 1 for exact criticality (\( \Delta = 0 \)). We see a well-formed attached undular bore upstream of the positive step, zero over the step itself, and a well-formed detached undular bore downstream of the negative step, in agreement with the theoretical predictions made above. This scenario has also been found in direct numerical simulations of the Euler equations for free surface flow over a step by Zhang & Chwang (2001).

References


Korteweg, D.J. and de Vries, H. 1895 On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. \textit{Philosophical Magazine}, \textbf{39}, 422-443

Lord Rayleigh 1876 On waves. \textit{Phil. Mag.} \textbf{1} 257-279


Russell, J.S. 1844 Report on Waves. \textit{14th meeting of the British Association for the Advancement of Science}, 311-390

