Transport of Vorticity through a Finite Channel

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Abstract:
The paper addresses the plane inviscid incompressible flows through a finite duct. The boundary conditions are: the normal velocity is given at both inlet and outlet cross-sections, and in addition vorticity is given at the inlet. Attention is focused on effects of these boundary conditions on dynamics of vorticity. We study evolution of perturbations of steady flows and show that at there are least three qualitatively different possibilities: first, the perturbations can be washed out the duct completely in a finite time; second, they can be trapped in the duct and then form some steady vortex structures; third, they can develop themselves into an unsteady pulsating flow.

Key-words:
vortex dynamics, inviscid fluid, vortex breakdown

1 Introduction

Wang & Rusak (1997) and Rusak, Wang & Whiting (1998) put forward an important concept of incipient vortex breakdowns based on the consideration of inviscid swirling flows in a finite circular pipe. Their analysis involves the pipe finiteness essentially, and therefore, there is a natural question concerning effects of boundary conditions which are imposed on the flow inlet and outlet. Gallaire & Chomaz (2004) considered such a question and, among other results, pointed out that under certain boundary conditions, the kinetic energy turns out to be non-increasing for every solution of the linearized equations despite they considered an inviscid fluid. In physical terms one can say that there is an effective dissipation; indeed, every inviscid fluid flow through a finite duct or pipe always represents a generic non-conservative system; the violation of the conservation laws takes place because there is pumping in and withdrawal of energy and vorticity due to the inflow and outflow of the fluid. The detailed balance depends on the boundary conditions.

The example discussed above represents a particular case of the withdrawal domination, which often reveals itself via existence of non-increasing Liapunov functionals. Existence of them makes impossible any instability; in particular, the decreasing of the kinetic energy makes impossible the incipient vortex breakdowns. However, additional investigations is necessary to decide whether or not every perturbation decay completely. This is especially subtle problem if we pass to finite perturbations. In the present communication we study this problem for the simplest available flow model. Namely, we restrict ourselves with two-dimensional (plane) flows of an inviscid incompressible fluid through a rectilinear duct. The boundary conditions are: normal velocity of fluid is prescribed everywhere at the boundary and vorticity is given at the inlet. We consider merely steady boundary data; in particular, we require a steady inflow of fluid through one its side (inlet) $S^+$ and steady outflow through the opposite side (outlet) $S^-$. Morgulis & Yudovich (2002) gave detail analysis for the competition and balance between the
withdrawal and ‘pumping-in’ processes for such a problem. They considered both the linearized and the exact non-linear equations and found that for some classes of flows the withdrawal dominates over the pumping-in; this dissipation is described in terms of some quantities which represent integrals of motion for the flows in an infinite duct but happen to be monotonically decreasing for some flows in finite ducts. In addition, they proved that under certain additional conditions the perturbations do decay completely with time (i.e. some classes of steady flow possess the asymptotic stability). However, the stability results have been established within the linear approximation only.

In this communication we consider behaviour of finite perturbations for some simplest types of flows. First, we consider the boundary data which admit the decreasing Liapunov functional. In such a situation sufficiently small perturbations are washed out the duct in a finite time. However, more strong perturbations are trapped in the duct i.e. they extend their stay in the duct to an uncertainly long time. On late stages of their evolution they probably tend to form a steady vortex structure with massive recirculation domains, which accumulate the trapped vorticity. In addition, we study a completely different situation where the boundary conditions produce an unstable steady flow. Then even the small perturbations can grow downstream and give rise to an unsteady oscillating vortex structure.

2 The Governing Equations.

Consider a two dimensional (plane) flow of an inviscid incompressible and homogeneous fluid. The flow domain is $D = \{(x, y) : 0 < x < l; 0 < y < 1\}$. Advection of vorticity is described by equations

$$\omega_t + \psi_y \omega_x - \psi_x \omega_y = 0; \quad -\Delta \psi = \omega,$$

where $\omega = \omega(x, y, t)$ is vorticity and $\psi = \psi(x, y, t)$ is stream function. The boundary conditions are

$$\psi|_{y=0} = 0, \quad \psi|_{y=1} = Q, \quad \psi|_{x=0} = \psi^+, \quad \psi|_{x=l} = \psi^-;$$

$$\omega|_{x=0} = \omega^+. \quad (3)$$

Here $\omega^+, \psi^+$ and $\psi^-$ are given time-independent functions of $y, 0 < y < 1; Q \equiv \text{const} > 0$ is the total flux of fluid through the duct. In addition, we assume that both $\psi^+$ and $\psi^-$ are the monotonically increasing functions. Consequently, in all further considerations the inlet coincides with that side of the rectangle where $x = 0$, while the outlet coincides with the opposite side $x = l$, so that the condition (3) prescribes the vorticity at the inlet.

In the sequel we restrict ourselves with the simplest boundary data:

$$\omega^+ \equiv 0; \quad \psi^+ = \psi^- = Qy, \quad Q > 0. \quad (4)$$

Evidently, these data produce a steady uniform flow $\Psi_0 = Qy$ which is irrotational. A vortical initial perturbation results in the unsteady flow which admits a decreasing Liapunov function

$$I(t) = \int_D \omega^2(x, y, t) \, dx \, dy, \quad \text{where} \quad \frac{dI}{dt} = -Q \int_0^1 \omega^2(l, y, t) \, dy. \quad (5)$$

In fact, the decreasing Liapunov functionals exist for rather wide classes of the boundary data (see Morgulis & Yudovich (2002)). However, they do not have the form (5). Instead, one have to apply the functionals introduced by Arnold (1966).
Figure 1: Duration $T$ of stay in the duct for the perturbations of the uniform flow ($Q=0.03$). The initial perturbations are given in (6) where $k_x = k_y = 1$ (clause (a)), $k_x = 1; k_y = 2$ (clause (b)), $k_x = k_y = 2$ (clause (c)), and $k_x = 2; k_y = 3$ (clause (d)).

3 Results.

Our study of the finite perturbations are based on the numerical solution of the problem (1)-(2)-(3). The numerical scheme is the vortex method. We omit the details because this numerical scheme is presented separately at the Congress.

The problem was solved for the duct length $l = 3$. The initial vorticity was

$$\omega_0(x, y) = A \sin(k_x \pi x/l) \sin(k_y \pi y).$$

The first series of computations were performed to examine how long the perturbations can stay in the duct depending on their initial amplitudes given in (6). The results are presented in Fig.1. One can see that the ‘lifespan’ of perturbation grows drastically near some critical amplitude which represent a threshold for transition from the washing out to the trapping of perturbations. Notice that for the multi-pole perturbations (c) and (d), the critical amplitudes is always greater than those of dipole (b) or monopole (a), and the dependence $T(A)$ is more complicated.

The next question is whether or not a non-separated flow (say, uniform one) can coexist with some steady, stable, and separated flows which obeys the same boundary conditions? The observations of the trapping suggest the positive answer. At least, the trapped vorticity seems to be steady in the large scale, but the small scale motions form an unsteady ‘atmospheres’ around almost steady vortex cores. To get rid of the small scale noises, we stop the solving of the unsteady problem and then perform a filtration of the instant vorticity field. This refined vorticity field is taken as new initial data, with which the solving of the unsteady problem is recommenced. After a few iterations, we obtain such initial data that provide very fast evolution to a steady state. In this way we obtained several steady states which coexist with the uniform flow. They are presented in Fig.2. The pairs stream function-vorticity found numerically form regular curves with very good accuracy (see Fig. 3). Qualitatively similar results have been obtained for several other flow which admit the decreasing Liapunov functionals.

\[1\text{ author: V. Govorukhin, title: Numerical analysis of ideal fluid flows through plane duct of finite length.}
\[2\text{name ‘separated’ is referred to flows with recirculation domains in which the material particles stay forever.}
\[3\text{Such pairs must form a curve for every steady two-dimensional inviscid flow.}
Next we attempt to answer the question: what happens if an unstable flow profile is prescribed at the inlet? For this purpose, we consider an 1-parametric family of the boundary data

$$
\psi^+(y) = \psi^-(y) = Qy, \quad \omega^+(y) = A_b(1 + \lambda(2\psi^+(y)/Q - 1)^2)^{-1}.
$$

Here $\lambda > 0$ is fixed while $A_b > 0$ is considered as the family parameter. The corresponding profile of the vorticity-stream function dependence is displayed in Figure 4(b). Thus, the inlet vorticity concentrates itself at the point $y_0$, such that $\psi^+(y_0) = Q/2$. For the large values of $A_b$ and $\lambda$ the vorticity profile tends to a vortex sheet, so that the shear layer instability is expected. If $A_b = 0$, the data (7) produces the uniform flow with velocity $Q$. Regarding relatively small $A_b > 0$, one result of Morgulis & Yudovich (2002) states that for every fixed $\lambda > 0$, there exists $a > 0$ such that for every $A_b < a$ the data (7) produces a non-separated steady flow, possessing the asymptotic stability (in the linear approximation).

In order to find the steady solutions for small $A_b$ we solve the unsteady problem (1)-(3) with the boundary data (7), where $A_b = a_1$, $0 < a_1 << 1$. As the initial condition we chose the uniform flow with velocity $Q$. Then the resulting unsteady flow quickly evolves to a steady state, which corresponds to $A_b = a_1$. This new steady flow is taken as an initial state for a slightly larger value of $A_b$, and the procedure is repeated. As a result we obtain an 1-parametric family of steady flows whose typical member is presented in Fig. 4. Note that the vorticity-stream function dependence found in this flow is exactly the same as at the inlet (see Figure 4(b)).
Continuing this family to higher $A_b$ we arrive at a threshold $A_{cr}$ such that the flow relaxes to a steady one, provided $A_b < A_{cr}$, and relaxes to an unsteady oscillating flow, if $A_b > A_{cr}$, see Fig. 5. Thus, the excess over the critical parameter value leads to an instability followed by an excitation of the secondary pulsating flow. (Let us remind that we study a non-conservative system, for which an excitation of self-oscillations (say, due to the Poincare-Andronov-Hopf bifurcation) is a generic phenomenon.) The threshold value $A_{cr}$ is approximately 0.45.

4 Conclusions

The observed abrupt transition from the washing out to the trapping of perturbations at the certain critical initial amplitudes meets an analogy in the dynamics of a material particle subject to both the Rayleigh friction and a conservative force. Let the force potential have two pits separated by a hillock. If a particle is located at the bottom of one pit, then after a light push it relaxes to the same equilibrium; however after a proper shove it may overcome the hillock and relax to another equilibrium. Of course, this is an utterly rough analogy but it highlights the principal fact underlying the trapping phenomena: the steady problem can have many solutions for the same boundary data. Amongst them, non-separated flows (if any) are generically isolated from the separated flows but seem to be not isolated one from another like in the conventional case of inviscid steady flows within wholly impermeable boundaries. After a sufficiently strong initial perturbation the flow is capable to evolve to a remote steady state. One can conjecture that similar non-linear mechanisms can give rise to a vortex breakdowns even in those flows which admit the decreasing Liapunov functionals (e.g. Gallaire & Chomaz (2004)). These observations put forward the question: which ‘intensity’ of perturbation is necessary to initiate the trapping and what kind of measure of this ‘intensity’ (e.g. energy, enstrophy, amplitude, or anything else) should be in use? For example, looking at Fig. 1, one can guess that (for a fixed amplitude) the short waves of vorticity are harder to trap than the longer ones. However, this issue remains almost uninvestigated. More general problem is to explain the non-uniqueness of steady states and predict their selection using the initial data. These are really hard issues. Perhaps, they can be approached within the cosymmetry theory (Yudovich (1995)), which successfully explains the similar features of convective flows in a porous media.
The example of self-oscillatory flow presented above sharply contrasts with oscillatory instabilities of plain Poiseuille or Taylor-Couette flows. Our example shows that self-oscillations of incompressible fluid can be generated and maintained merely by the proper inlet and outlet conditions with no viscous dissipation. Such a mechanism can be relevant to very high Reynolds numbers.

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References


GALLAIRE, F. AND CHOMAZ, J.-M. 2004 The role of boundary conditions in a simple model of incipient vortex breakdown, Physics of Fluids 16, iss. 2, pp. 274-286

