Non-linear Biot Darcy’s process of consolidation

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Abstract :

In this work we present the problem of creeping of the porous medium described by a linear and nonlinear poroelasticity models based on physical relation in the Biot’s body. The porosity function which depends on a dilatation of skeleton and fluid is introduced into the nonlinear model, and a process of medium deformation is also presented. The filtration in soil is calculated by the Finite Element Method for linear and nonlinear poroelasticity models, and the differences between these results are indicated. As a conclusion, some physical consequences of this type of nonlinearity in the Biot’s model are discussed.

Résumé :

Dans le présent article, nous cherchons à étudier le processus du fluage d’un milieu poreux qui est décrit par le modèle poroélastique linéaire et non-linéaire de Biot. En ce qui concerne le modèle non-linéaire, nous avons admis le changement de la porosité en fonction de la dilatation de la carcasse solide du sol et de la dilatation de l’eau dans le processus de la déformation du milieu poreux. Les calculs numériques ont été obtenus selon la méthode des éléments finis. Nous avons aussi essayé d’analyser l’influence de la non-linéarité adaptée au processus de la filtration d’un liquide à travers le milieu poreux.

Key-words :

Biot’s model, filtration, consolidation

1 Introduction

A permanent development of numerical methods, equipment, and software enables us to resolve an engineering problems based on more complex mathematics models. This allows us to obtain improving approximation of the real physical occurrences. Even simple rheology models are complicated in porous media mechanics. Thus, it is essential to search for compromise between compounded mathematic description of the assumed model in microscopic and macroscopic scale and the ability to obtain a solution. A rheology model of porous medium, which is usually simplified to a model of ideally elastic body (Hooke’s body), can serve as an example.

Laboratory research shows that the linear model of elasticity for porous medium is much too simplified, especially in scope pertaining to soil. Physical compounds that define processes before reaching the bordered values are nonlinear for soil medium. Framework of soil medium comprises separated grains or plates. Following forces may interact between them:

- force of static friction between each grain
- force of viscous friction, if a liquid phase (for example, in case of coherent soil, water with electrostatic forces) occurs between grains,
– internal forces present in the material cementing the grains of soil; such material, having different mechanics characteristic due to the chemical processes, can be obtained in the course of the filtration of mineralized water through the medium.

Deformation of medium due to load has a reversible character, when the movement is a result of elastic deformations of grains and or plates. In case the movement occurs locally due to overpassing the limits of the forces mentioned above, the process of deformation has a non-reversible character.

Mathematic models describing the process of stress-strain, including phenomena described above, would be for sure close to reality, but it would appear more complicated than the models usually applied to soil processes.

The purpose of this paper is to estimate the impact that the porosity deformed medium in consolidation process has on the Biot’s constants by using the numerical methods. The authors applied the convergence bi-scale method known from the theory of homogenization described by Auriault & Sanchez Palencia (1977), Auriault (1983), Auriault and the others (1990). The results of numerical calculations deformation of porosity medium based on introduced hereinafter relations were compared with the results of deformation of medium obtained on the basis of Biot-Darcey’s classic model where the constancy of the porosity factor is one of the basic assumptions.

2 Parameters of Biot’s model in function of medium porosity

In order to define relations between the factors in Biot’s-Darcy’s model and porosity following two separate numerical problems have been solved in two-dimensional system:

(i) Flow of the Newton’s liquid through non-deformamable porosity medium to define relation between Darcey’s permeability tensor and the porosity of skeleton,

(ii) Deformation of the elastic porous medium to define the relation between factors in Biot’s-Darcey’s model (that is the N modulus of shearing strain and the A modulus of bulky strain) and the porosity medium. Relations between the other Biot’s Q and R factors and the porosity were based on Fatt’s (1959) and Biot’s & Wilis’s (1957) researches.

A simple geometric shape of one periodic cell of squared shape medium with circle inside that represent the grain of framework was used for the purpose of numerical analysis. Change of porosity medium was obtained by changing the diameter of the circle.

It is obvious that in order to improve the accuracy of the analysis, a solid periodic cell model should be applied, as well as, the shape and the position of grains in the cell should be changed. However, the numerical analysis shows that a structure of the cell in micro-scale has only a small influence on the results. Therefore, it is stated that the application of simplified model is justified.

2.1 Dependency of Darcy’s permeability tensor to porosity

The permeability tensor in porosity function for a two-dimensional constant flow through the periodic cell with \( \Omega \) surface, bordered by \( \Gamma \) can be defined based on a Stoke’s equation and a continuity equation (see: Auriault (1986), Strzelecki and the others (2007)):

\[
\begin{align*}
- \frac{1}{\rho} \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &= 1, \quad x, y \in \Omega_f \\
- \frac{1}{\rho} \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) &= 0, \\
\nabla \cdot \vec{v} &= 0, \quad \vec{v} = (u, v)
\end{align*}
\]
with the boundary conditions in the contact of the liquid with a solid body:

\[ \bar{v}_i = 0, \quad \frac{\partial \rho}{\partial n}_{|v} = 0 \]  

(3)

and with the periodic conditions on the borders of a cell, where there is a contact liquid-liquid between neighbouring cells:

\[ [u] = 0, \quad [v] = 0, \quad [\rho] = 0. \]  

(4)

The problem defined by equation (1)-(4) above is impossible to be resolved by a numerical method, because the system of equations is singular, therefore we suggest to modify of this system by taking the divergence of the (1):

\[ \nabla \cdot \left( -1/\rho \cdot \nabla p + \mu \nabla^2 \bar{v} + \bar{g} \right) = 0, \quad \bar{g} = (1,0) \quad \text{in} \quad \Omega_f \]  

(5)

Starting with a known mathematic identity in equation (5) the Laplace’s operator should be replaced with \[ \nabla^2 \bar{v} = \nabla \left( \nabla \cdot \bar{v} \right) - \nabla \times \nabla \times \bar{v} \] and by use of (2) and the relation \[ \nabla \cdot \left( \nabla \times \nabla \times \bar{v} \right) \equiv 0 \] we receive:

\[ \nabla^2 p = 0 \]  

(6)

Equation (6) is replaced by the continuity flow equation (2), and a new closing boundary condition is introduced as a result of homogenization theory (Strzelecki and the others, 2007):

\[ \int_{\Omega_f} pd\Omega = 0 \]  

(7)

Problem (1)-(4) could be now resolved by the following:

\[ -\frac{1}{\rho} \frac{\partial p}{\partial y_1} + \mu \left( \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2} \right) = 1 \]

\[ -\frac{1}{\rho} \frac{\partial p}{\partial y_2} + \mu \left( \frac{\partial^2 v}{\partial y_1^2} + \frac{\partial^2 v}{\partial y_2^2} \right) = 0 \]

(8)

\[ \nabla^2 p = 0 \]

with the boundary conditions (3), the periodic condition (4) and the closing condition for a pressure function (7). A velocity field is a solution of this system. In accordance with the rules described by Strzelecki and the others (2007) the velocity field should be integrated by the area of the cell \( \Omega_f \), to determine the components of a permeability tensor \( \tilde{k}_{11}, \tilde{k}_{12} \). In order to define other components \( \tilde{k}_{21}, \tilde{k}_{22} \), we calculate the impact of an unit mass force along the \( y \) axis, and then the procedure of the calculation of components is being repeated.

The square cell with non-dimensional 2 length was applied in the course of numeral tests based on the finite elements method.
Calculations for different grain positions in the cell have been made for this model. The results of the central position of the grain in cell as well as finite element mesh are presented below on Fig. 1.

![Fig. 1 – a) Calculation area - finite elements mesh, b) isoline horizontal velocity of filtration](image1)

The permeability tensor was obtained in the course of the calculation. The tensor’s components of $\bar{k}_{ij}$, except for the main tensor’s diagonal, can be assumed to be equal to zero. Because of the tensor’s symmetry, values on diagonal are equal, therefore, it can be assumed that the permeability tensor is defined by the permeability factor $k$.

Variation of the filtration factor $k$, in the porosity function is presented on Fig. 2.

![Fig. 2 – Dependency of filtration factor to porosity](image2)

2.2 Relations of $N$ and $A$ Biot’s factors to porosity

The border issue of deformation process of a solid phase of the medium, based on Auriault’s studies (1986), comprises only a solution of following system for equations in micro scale:

$$\frac{\partial}{\partial x_j} (a_{ijk} \epsilon_{kh} (\bar{w})) = 0, \quad \bar{w} = (w_x, w_y), \quad x, y \in \Omega,$$

(9)
with the boundary condition and the periodicity condition:

$$\left[ a_{ijkl} e_{lih} \left( \vec{w} \right) \right]_{ijl} = 0$$

$$\left[ \vec{w} \right] = 0, \quad \left[ a_{ijkl} e_{lih} \left( \vec{w} \right) \right] = 0$$ (10)

where $\vec{w}$ is a displacement vector, $a_{ijkl}$ is a tensor of elasticity 4th range, $e_{lih}$ is a second range deformation tensor.

Based on the procedures of bi-scale convergence we assume that all the considered function depend on two variables defining macro and microscopic changes accordingly. In case of the two-dimensional problem we consider the process of a periodic $\Omega$ cell deformation caused by a unit deformation from a macroscopic scale in a flat state of deformation. Physical relations of the elasticity flat model can be expressed by the following equations in macroscopic scale.

$$\sigma_{xx} = 2\tilde{N}E_x + \tilde{A} \left( E_x + E_y \right) + 2\tilde{N} \frac{\partial w_x}{\partial x} + \tilde{A} \left( \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} \right),$$

$$\sigma_{yy} = 2\tilde{N}E_y + \tilde{A} \left( E_x + E_y \right) + 2\tilde{N} \frac{\partial w_y}{\partial y} + \tilde{A} \left( \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} \right),$$

$$\sigma_{xy} = 2\tilde{N} \left[ E_{xy} + \frac{1}{2} \left( \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} \right) \right].$$ (11)

where $E_x$ – a relative extension along the $x$ axis, $E_y$ – a relative extension along the $y$ axis, $E_{xy}$ – a shearing strain.

The equilibrium system of equations has a following form:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0,$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0.$$ (12)

where $\tilde{N}, \tilde{A}$ are the Biot’s constants in the case when the medium does not have porous, that means when it is a continuous body. The system above needs to be compliant with the boundary conditions and the periodic conditions with a closing integral condition:

$$\int_{\Omega_x} \tilde{w}_x d\Omega_x = 0, \quad \int_{\Omega_y} \tilde{w}_y d\Omega_y = 0$$ (13)

Calculation of the searched factor values are obtained by assuming $E_x = 1, E_y = 0, E_{xy} = 0$.

The analogical solution when $E_x = 0, E_y = 1, E_{xy} = 0$ can be obtained assuming the symmetry of an area geometry.

We get the searched values $2N + A$ and $A$ by the determination of an average value of components of $w_x$ and $w_y$ of the displacement vector.
Distribution of displacement in x and y axis’ direction is presented on Fig. 2 in the scale of a periodic cell.

Figure 4 contains function of parameter N and M depending on porosity of medium.

The process of identification of relations between other Biot’s parameters and porosity of medium requires the solution of more complicated system of equations in the microscopic scale. However, provided that water is a liquid medium, we can assume that the fluid phase is less compressible than skeleton of porous medium.

The R parameter which defines relations between liquid dilatation and pressure of fuzzy liquid relates to porosity in linear way.

With reference to studies of Biot’s & Willys (1959) and Emmerich (1984) it can be stated that the following simple relations occur between Biot’s H and R parameters and porosity f:

$$H = \frac{R}{f}$$

(14)
3 Impact of non-linear model on creeping medium process

Two issues of dam and subsoil creeping under their dead-weight and water load accumulated in reservoir have been solved in order to define the impact of non-linearity in the Biot’s model. First, based on the classical linear Biot’s model and the second for calculated relations between Biot’s parameters and porosity (non-linear model).

Calculations enable us to compare all important values: stress, strain and velocities of filtration. All range of differences between the linear and non-linear model cannot be discussed herein due to the limited volume of this paper. Thus, only the comparison of the vertical strain and plasticity potential Columb’a-Mohr’s are presented on Fig. 5 and Fig. 6 accordingly.

![Fig. 5 – Isoline of vertical displacement a) linear model b) non-linear model](image1)

![Fig. 6 – Isolines of plasticity Coulomb’s-Mohr’s potential for: a) linear model b) non-linear model.](image2)

As a result of deformation process, the primary homogeneous medium becomes a non-homogeneous medium with effective factors of model values different from initial values. Fig. 7a presents change of the porosity medium. Distribution of value of \( N \) factor in the consolidation area is presented in Fig. 7b.)
FIG. 7 – a) Isolines of porosities in consolidation medium after creeping process,  b) Isolines of effective parameter value $N$ after consolidation

4 Conclusions

Based on the foregoing numerical experiments it is stated that despite of the fact that the model is not linear, results of calculations are consistent with engineer’s intuition which is based on classical linear Biot’s model. Only the physical values calculated in creep of soil are different. The character of curves obtained in the calculations is also the same both in linear and non-linear model.

It is obvious that the differences in values of stress and strains would be more significant in the case of the stratified medium of soil.

This paper is one of the stages of a complex analyses of nonlinear filtration and consolidation process.

References


Auriault J.-L. 1986 Mécanique des Milieux Poreux Saturés Déformables, Cours de 3 ème cycle *MMGE* Grenoble pp. 1–71


Emmrich R. 1984 Experimental verification of electro-kinetic consolidation model (PhD dissertation in Polish), Wroclaw University of Technology’s Institute of Geotechnics, *Report ser. PRE No. 307*

Fatt I., Willis 1959 The Biot-Elastic Coefficients for a Sandstones, *ASME.*