One-dimensional consolidation issue of the porous medium with the rheological Kelvin-Voigt skeleton

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Abstract:

In this paper, the analytical solution of porous medium consolidation with the rheological Kelvin-Voigt skeleton is presented. The rheological model is a model which elements are four basic physical features: elasticity, viscosity, plasticity and strength. One-dimensional problem insist on solving equations for porous column filled with liquid and being a subject of one-dimensional compression with load through porous plate (allowing fluid flow), pressure gradient and weight of column itself. Results obtained may be used also for determination of effective parameters of the Biot model. According to the types of equations, in the range of analytical solutions we will make a use of techniques based on double integral transformation of Laplace and Fourier. Within the range of boundary issues solutions of porous media consolidation the use will be made of a finite element method.

Résumé:

Dans cet article traitant de la consolidation du milieu poreux, nous présentons les résultats des calculs analytiques du modèle de Kelvin–Voigt. Ce modèle rhéologique se définit par quatre attributs physiques: elasticité, viscosité, plasticité, et résistance. L’objet de l’analyse consiste à résoudre l’équation d’une colonne poreuse remplie de liquide et soumise à la compression monodimensionnelle par la charge d’un plat poreux, le gradient de pression hydrostatique et le poids de la colonne elle-même. Les résultats ainsi obtenus pourront être utiles dans la détermination de paramètres efficaces pour le modèle de Biot.

Key-words: Biot model, analytical solution, rheological model

1. Introduction

During consideration of porous medium consolidation problem, it would be perfect to obtain a solution of a boundary issue in analytical form. In such a case, there exist a possibility of carrying out any analysis of the process being investigated with mathematical analysis tools. In most cases however, boundary issue is too complicated with regard to irregularity of investigated area, irregularity of source functions, complicated form of partial differential equations and therefore obtaining an analytical solution is impossible. In such cases, numerical methods are used for obtaining approximate solutions.

In this paper, a method for obtaining analytical solutions, with the use of Laplace transformation is presented using the example of displacements in Biot poroelasticity model with rheological Kelvin-Voigt skeleton. The Biot-Darcy poroelasticity model is introduced with equations for movement of liquid and solid medium phase, flow continuity equation and constitutive relations which allows to write a collective equation set for a Biot-Darcy linear consolidation theory for isothermal process describing displacements of the skeleton and tension in liquid. Next, obtained solution were a subject of comparison with a numerical solution.
Assuming, according to Strzelecki et al. (2007) constitutive relations for isothermal process in form:

\[
\begin{align*}
\sigma_{ij} &= 2N\Psi_k\epsilon_{ij} + (A\epsilon + Q\theta)\delta_{ij} \\
\sigma &= Q\epsilon + R\theta
\end{align*}
\]  

(1)

The set of equations for porous medium consolidation with rheological Kelvin-Voigt skeleton for quasistatic processes, with the use of Einstein index notation, might be written in form:

\[
\begin{align*}
N\Psi_k \nabla^2 u_i + (M + N\Psi_k + MP_k)\epsilon_{\sigma_i} &= -\frac{H}{R}\sigma\dot{\epsilon}_i \\
\frac{k}{f^2} \nabla^2 \sigma &= \frac{1}{R} \dot{\sigma} - \frac{H}{R} \dot{\varepsilon}_i
\end{align*}
\]

(2)

Where: \( N, M, H, R \) – Biot constants; \( T = \frac{\eta_s}{N} \) – viscosity parameter; \( \Psi_k = 1 + T \frac{\partial}{\partial t} \) – differential operator in Kelvin-Voigt skeleton; \( k \) – Darcy filtration coefficient; \( f \) – skeleton porosity. \( \eta_s \) – form viscosity of skeleton. \( \epsilon_{\sigma_i} \) – velocity of skeleton dilatation, \( \theta \) – velocity of fluid dilatation, \( u \) – skeleton displacement, \( \sigma \) – fluid stress, \( \sigma_{ij} \) – skeleton stress, \( \delta_{ij} \) – Kronecker delta.

\[
\begin{align*}
N\nabla^2 u_i + (M + N)\epsilon_{\sigma_i} + NT\nabla^2 \dot{u}_i + (N + M)\theta \dot{\varepsilon}_i &= -\frac{H}{R}\sigma\dot{\epsilon}_i \\
K\nabla^2 \sigma &= \frac{1}{R} \dot{\sigma} - \frac{H}{R} \dot{\varepsilon}_i
\end{align*}
\]

(3)

The above equation set describes the process of consolidation caused by filtration flow of viscous Newtonian liquid filtrating through porous of Kelvin-Voigt skeleton. Solutions of this equation set were the research subject of: Auriault et al. (1981); Emmrich (1984), Strzelecki & Żak (1980).

2. Preliminary assumptions of one-dimensional consolidation model with the Kelvin-Voigt rheological skeleton

Below, results of consolidation process analysis of column-shape porous medium, which solid particles are subject to load and hydrostatic pressure gradient. Examples of such medium might be cohesive soils built of secondary minerals such as illite, montmorillonite and kaolinite. The issue of analytical solution of a one-dimensional problem with the classical Biot model were the research subject of: Auriault et al. (1981), Jasiewicz (1968), Bauer & Strzelecki (1980), Gaszyński (1984), Emmrich (1984) and Derski (1967, 1969). As mentioned above, we assume that all kinds of loads are applied instantly at time \( t = +0 \) which is represented by introducing a Heaviside function. The subject of analysis was the consolidation caused by external load and hydrostatic pressure gradient which schematically is presented on Fig. 1.
Boundary condition:

Load condition on the upper boundary: \( \sigma_{33}(h,t) = -P \eta(t) \)

Fluid stress condition on the upper boundary: \( \sigma(h,t) = -p_s \eta(t) \)

Stress condition on the bottom boundary: \( \sigma(0,t) = -p_b \eta(t) \)

Displacement condition on the bottom boundary: \( u(0,t) = 0 \)

Initial condition: \( \sigma^{(0)} - H\epsilon^{(0)} = 0 \)

Functions: fluid stress, skeleton stress and deformation were the subject of Laplace transformation. Image functions in Laplace space are denoted:

\[ \tilde{\sigma}_{33}, \tilde{\sigma}, \tilde{u}, \tilde{\epsilon}, \tilde{\theta} = L(\sigma_{33}, \sigma, \epsilon, \eta, \theta) \]

### 3. Analytical solution:

Taking into account the initial condition, the equation set for porous medium consolidation with Kelvin-Voigt skeleton in Laplace space takes a form:

\[
\begin{cases}
(M + 2N + 2NTs) \frac{\partial^2 \tilde{u}}{\partial x^2} = -\frac{H}{R} \frac{\partial \tilde{\sigma}}{\partial x} \\
k \frac{\partial^2 \tilde{\sigma}}{\partial x^2} = \frac{s}{R} \tilde{\sigma} - \frac{H}{R} \frac{\partial \tilde{u}}{\partial x}
\end{cases}
\]

In order to obtain Laplace transformation, for boundary conditions, the transformation of Heaviside function were performed, thus boundary conditions in Laplace space takes form:

\[ \tilde{\sigma}_{33}(h,t) = -\frac{P}{s}, \quad \tilde{\sigma}(h,t) = -\frac{P_a}{s}, \quad \tilde{\sigma}(0,t) = -\frac{P_b}{s}, \quad \tilde{u}(0,t) = 0, \quad \text{where } s \text{ is the transformation parameter.} \]

Performing transformations of equation set (4), we obtained:

\[
\frac{\partial^3 \tilde{\sigma}}{\partial x^3} = p(s) \frac{\partial \tilde{\sigma}}{\partial x}, \quad \text{where: } \quad p(s) = \frac{s(a + s)}{b(c + s)} = \frac{\frac{H^2 + R(M + 2N)}{2NT} + s}{KR \left( \frac{M + 2N}{2NT} + s \right)}
\]

The solution of equation set (5) in Laplace space, according to the work of Ditkin & Prudnikow (1964) is function:

\[ \tilde{\sigma} = Ae^{x\sqrt{p(s)}} + Be^{-x\sqrt{p(s)}} + C \]

After double differentiation, substituting solution into flow equation in equation set (4) and double integration we obtain image function of displacements:

\[ \tilde{u} = -\frac{H}{2RNT \sqrt{p(s)}(c + s)} \left( Ae^{x\sqrt{p(s)}} - Be^{-x\sqrt{p(s)}} \right) + Dx + E \]

were: \( D, E \) are functions of integration parameter \( s \).

Substituting obtained functions (6) and (7) into boundary conditions, utilizing the relation \( C = HD \) and constitutive relations, we obtained algebraic equation set that was used for determining constants \( A, B, C, D \) and \( E \). After substituting constants into image functions (6) and (7) we obtained final image forms of equation (6) and (7). In this paper we present an analysis for image function of displacement \( \tilde{u} \):
To find retransformed form of displacement image function, the residue theory was used. Based on Cauchy residue theorem and Jordan lemma, the original of rational function $\tilde{F}(s)$ with single poles $s_k$ is rational function with the form of:

$$L^1[\tilde{F}(s)] = \frac{LI(0)}{MI0} + \sum_{k=1}^{n} \frac{LI(s_k)}{s_kMI'(s_k)} e^{sp}$$

Where $LI(s)$ and $MI(s)$ are prime polynomials with respect to themselves and degree of $LI(s)$ is lower than of $MI(s)$. After retransformation, the function describing displacements $u$ takes a form of:

$$u = \frac{H^2P}{2hRN^2T^2} S_1 + \frac{H^2P}{2hRN^2T^2} S_2 - \frac{H^2(R + H)}{2hR^2N^2T^2} S_3 - \frac{H}{hRNT} S_4 + \frac{x(P_a - P)}{2NTa} \left[1 - e^{-s} \right]$$

(10)

Containing series:

$$S_1 = \sum_{k=1}^{n} \frac{(-1)^n + 1}{s_k P'(s_k)(c + s_k)(a + s_k)} e^{sp}, S_2 = \sum_{k=1}^{n} \frac{(-1)^n \cos \left( \frac{n\pi}{h} x_3 \right) + 2}{s_k P'(s_k)(c + s_k)(a + s_k)} e^{sp}$$

$$S_3 = \sum_{k=1}^{n} \frac{P_a(-1)^n + P_b}{s_k P'(s_k)(c + s_k)(a + s_k)} e^{sp}, S_4 = \sum_{k=1}^{n} \frac{P_a(-1)^n + P_b}{s_k P'(s_k)(a + s_k)} e^{sp}$$

(25)

3. Results of skeleton displacements calculations

Equation (10) i.e. the analytical solution were used for vertical displacements calculations $u$ for given loads: $P=1,5*10^5$ Pa; $P_a=0,55*10^5$ Pa; $P_b=1,2*10^5$ Pa; sample parameters $h=10,0$m; $f=0,35$ and Biot constants: $M=5*10^7$ Pa; $N=2,5*10^7$ Pa; $R=1,5*10^7$ Pa; $H=2,25*10^7$ Pa; $Q=3,75*10^7$. Results of calculations are presented on Fig. 2.
In order to verify consolidation results of poroelastic Biot medium with rheological Kelvin-Voigt skeleton, numerical investigation on one-dimensional of rheological Kelvin-Voigt model were performed with following boundary conditions: upper boundary: $\sigma = 1.5 \times 10^5$, $P_a = 1.5 \times 10^5$; bottom boundary: $P_b = 1.5 \times 10^5$.

The sample is rigidly restricted from bottom excluding vertical displacements of the bottom of the sample and is a subject of hydrostatic pressure gradient (caused by loads $P_a$ and $P_b$). Calculations results, performed with FlexPDE program, are presented on Fig. 3.

Presented plots illustrate the influence of pressure gradient in liquid on the creeping process of the sample with the analytical (Fig. 2) and numerical (Fig. 3) methods. Shapes of consolidation progress obtained with both methods are correspondent which confirms that equations obtained with analytical method do not contain mistakes. Small differences of obtained values result from numerical errors.

Presented plots, are significantly different from classic Biot model, in which immediate settlements occur and might often comprise significant fraction of final settlement. Numerous researches performed in edometers demonstrate, that in reality we do not observe immediate settlements of samples. Biot-Darcy model with Kelvin-Voigt skeleton describes the creeping process of the cohesive soils.

4. Summary and conclusions

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After appropriately long time, the creeping process trends to constant values of displacements, which means that the sample was consolidated. The final effect of consolidation is linearly proportional to cross-section coordinate of the sample.

Important influence on the consolidation computation with the numerical method have assumed initial conditions. The assumption of initial settlements different than zero, considerably disturb the computing process in utilized by us FlexPDE program and generate significant numerical errors.

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