ESTIMATION OF SOURCE LOCATION AND SPECTRAL PARAMETERS
WITH ARRAYS SUBJECT TO COHERENT INTERFERENCE

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RESUME

Cet article traite le probleme de l'utilisation d'un
reseau de M sensors. Le reseau de sensors, dont la
géométrie est arbitraire, est utilisé pour localiser la
source et estimer les paramètres du spectre où le
champ du bruit est dominé par une interférence
spatiallement cohérente.

Les expressions sont obtenues pour la matrice Fisher
d'information de la source spatiale et des paramètres
estimés du spectre et pour son inverse, la limite
Cramer-Rao sur les erreurs d'estimation. La source,
interférence et le bruit de fond sont supposés être
un processus de hasard gaussien dont la moyenne est
nulle, et indépendants statistiquement les uns des
autres.

Les entrées de la matrice Fisher d'information sont
exprimées en fonction des quantités physiques
suivantes: la forme du rayon convectionnel (le
conventional beam-pattern) et ses dérivées spatiales,
la forme nulle (a null-pattern), le nombre des
sensors M et les rapports de signal/bruit de la
source et de l'interférence. En conséquence certains
résultats sont obtenus concernant la corrélation entre
l'estimation des paramètres de la source spatiale et
l'estimation des paramètres du spectre, et la manière
dont la présence de l'interférence influence la
précision de l'estimation de la localisation de la
source, avec des paramètres de source spatiale connus
ou inconnus.

SUMMARY

This paper deals with the problem of using an M
sensor array of arbitrary geometry for source location
and spectral parameters estimation, when the noise
field is dominated by a spatially coherent
interference.

Formal expressions are obtained for the Fisher
information matrix (FIM) of the source spatial and
spectral estimated parameters and for its inverse, the
Cramer-Rao lower bound (CRLB) on the estimation
errors. Source, interference and background noise are
assumed to be zero mean Gaussian random process
statistically independent of each other.

The FIM entries are expressed in terms of physical
quantities: the conventional array beam-pattern and
its spatial derivatives, a null pattern, the number of
sensors M and the source and interference
signal-to-noise ratios. Results are obtained concerning
the correlation between source spatial and spectral
parameters estimates and the effect of interference on
the accuracy of the source location estimation. Both
known or unknown source spectral parameters are
considered.

1. INTRODUCTION

The use of sensor array to estimate source location
and spectral parameters has been studied extensively
during the last years [1]. If an array of M sensors in
arbitrary geometry is used and the received signal at
each sensor consists of a random signal plus
uncorrelated noise, the data covariance matrix at each
frequency, \omega, is given by

\begin{equation}
K(x)=K_0(x)K_0^T
\end{equation}

Under the assumption that the observation time T is
large compared with the correlation time of signal and
noise (T >> 1), Fourier coefficients associated with
different frequencies are uncorrelated. For a point
source (spatially coherent), the signal covariance matrix is

\begin{equation}
K_0(x)=S(x)y(x)y^*(x)
\end{equation}

where \text{S}(x) is the power spectrum of the transmitted
source and \text{y}(x) is a steering vector

\begin{equation}
y(x)=\exp(i2\pi f_1 t_1), \ldots, \exp(i2\pi f_N t_N)\end{equation}

In this case, the estimation problem has the key feature
that the asymptotic estimation errors for source
location parameters (bearing and range) are
uncorrelated with estimation errors for parameters
describing signal and noise spectra. This can be seen
by studying the Fisher information matrix (FIM), which is
the inverse of the Cramer-Rao lower bound (CRLB) on
the estimation errors of the unknown parameters:

Assuming that \text{K}_0(x) in (1) is known, the vector of
unknown parameters, \text{\theta}, can be decomposed into
\text{\theta} = \text{F}_0 \text{\phi}, where \text{\phi} is the vector of
differential delays, \text{\phi} = (t_1, t_2, \ldots, t_M)^T,
contains all spatial
unknown parameters and \text{F}_0 is a vector of the spectral
parameters of the source (any parametization of \text{S}(x)).

The FIM of \text{\theta}, \text{F}_0 can be partitioned as

\begin{equation}
\text{F}_0 = \begin{bmatrix}
\text{F}_{\text{\phi}} & \text{F}_{\text{\phi} \text{\theta}} \\
\text{F}_{\text{\theta} \text{\phi}} & \text{F}_{\text{\theta} \text{\theta}}
\end{bmatrix}
\end{equation}

For the case where \text{K}_0(x) is a diagonal matrix, it
can be shown that \text{F}_{\text{\theta} \text{\theta}} = \text{\phi}^T \text{F}_{\text{\phi} \text{\phi}} \text{\phi}
so the CRLB(\phi) = \text{F}_{\phi} \text{\phi} \text{F}_{\phi}^T and the
estimation errors of the spatial and spectral
parameters are asymptotically uncorrelated.

Furthermore, if in addition \text{K}_0(x) = \text{N}(\omega) \text{I}, where \text{I} is an
\text{M} \times \text{M} unit matrix, then

\begin{equation}
F_0 = \text{K}_0(x) \text{N}(\omega) \text{I}
\end{equation}

where

\begin{equation}
\text{K}(\omega) = \frac{2 \pi N(\omega)}{1 + \text{N}(\omega)} \quad \text{and} \quad \text{N}(\omega) = \text{F}_{\phi} \text{F}_{\phi}^T - \text{J}_{\phi} \text{M}_{\phi} \text{J}_{\phi}^T
\end{equation}

is an (M-1) vector of all ones [2]. Notice that here
\text{F}_{\phi} is not a function of \text{\phi}. The FIM (or the CRLB) for
any spatial parameter vector \text{\phi}, (bearing in the far
field case or range in the near field case) can then be obtained using [3]

\begin{equation}
F_{\phi} = \text{F}_{\phi}^T \text{F}_{\phi}^T \phi \text{J}_{\phi} \text{J}_{\phi}^T
\end{equation}

Kirklin and Dewey [4] showed that for the general case of
a non-diagonal noise covariance matrix, \text{F}_{\phi} becomes a
function of \text{\phi}. However, they did not consider \text{F}_{\phi} and
\text{F}_0, i.e. - S(x) assumed to be known.
In this paper we study the FIM for the case of a noise field dominated by a spatially coherent interference. This study provides some insight into the general problem of non-diagonal noise covariance matrix together with specific results for the important case of source parameters estimation in the presence of a spatially coherent interference.

To minimize algebraic complexity we give results only for the case of far field source and interference (location specified by bearings \( \phi \) and \( \beta \)). We assume that each radiates a narrowband, zero mean Gaussian random signal in the same frequency band (around \( \omega_o \)). The source and interference are uncorrelated and have spectra of levels \( S \) and \( I \) respectively, flat over the band \( W \). We also assume that the location and power of the interference, together with the noise level \( N \), are known so the vector of unknown source parameters is simply \( \mathbf{y} = (\omega_o, S)^T \). Under the narrowband assumption, the FIM is given by

\[
\mathbf{F}_\beta = \frac{\mathbf{W} \mathbf{J} \mathbf{W}^T}{\pi} \tag{8}
\]

where \( \mathbf{J} \) is the FIM for \( \beta \) when only Fourier coefficients associated with one frequency are available as data. (The more general case of unknown \( \phi \) and \( I \) is discussed in [53].)

2. THE FISHER INFORMATION MATRIX

For \( \theta = (\omega_o, S) \), \( \mathbf{J} \) is a \( 2 \times 2 \) matrix given by

\[
\mathbf{J} = \begin{bmatrix}
J_{\omega_o} & J_{S}
J_{S} & J_{S}
\end{bmatrix} \tag{9}
\]

For any zero mean Gaussian vector, \( \mathbf{J}_{\beta} \) is given by [6]:

\[
J_{\beta} = \text{trace} \left[ \mathbf{K}_{\beta} \mathbf{K}_{\beta} \right] \tag{10}
\]

where in our case \( \mathbf{K} \) in (1) is given by (2) and (3) at \( \omega = \omega_o \) and \( \mathbf{K} \) is given by

\[
\mathbf{K}_{\beta}(\omega_o) = \mathbf{J}_{\omega_o} \mathbf{Y}(\omega_o) \mathbf{Y}^T(\omega_o) + \mathbf{N}(\omega_o) I
\]

\( y \) is the interference steering vector given by

\[
y(\omega_o) = \{1, \exp(j \omega_o \bar{\theta}), \ldots, \exp(j \omega_o \bar{\theta} M)\}^T \tag{12}
\]

\( \bar{\theta}_m, m = 2, \ldots, M \) are the differential delays of the interference to the first (reference) sensor. Introducing

\[
Z = \frac{\mathbf{J}_{\beta}^{1/2}}{N^{M/2}} \quad \text{and} \quad \zeta = 1 - Z \tag{13}
\]

we notice that \( Z \) is the height of the normalized beam-pattern (BP) for a beam steered in the direction of the source \( \phi \), when the interference is incident from direction \( \beta \) (or vice versa). Also define

\[
y = \begin{bmatrix} y_{\omega_o} \ z_{\beta} \end{bmatrix}^T \ \text{where} \ \mathbf{U} = \begin{bmatrix} y_{\omega_o} \ z_{\beta} \end{bmatrix}^T \tag{14}
\]

where \( \mathbf{U} = \text{diag}(y) \); \( J_\eta = \text{diag}(\bar{\mathbf{U}}_\beta) \), \( \eta = 2, \ldots, M \). For a far field source, \( \bar{\mathbf{U}}_\beta = (\bar{x}_m \sin \phi + \bar{y}_m \cos \phi)/\sqrt{c} \) where \( (\bar{x}_m, \bar{y}_m) \) are the coordinates of the \( m \)-th sensor. Here we notice that \( Y \) of (14) is the output power of the system of Fig. 1 due to an input signal (interference) at bearing \( \beta \). This system nulls the source (at bearing \( \phi \)). Its output power is simply due to the interference. (When the source and interference coincide, \( Y = 0 \)). The elements of \( \mathbf{J} \) are given by:

\[
J_{\omega_o} = \frac{M^{2}(1+M \rho^2)\bar{Z}}{N^{2} \rho^2} = \frac{M^2}{N^2 (1+M \rho^2) r^2 + 2M \rho^2 r^2} \tag{15}
\]

and

\[
J_{S} = J_{\omega_o} - \frac{M \rho^2}{N \rho^2} \tag{15a}
\]

and \( J_{\omega_o} = J_{\omega_o} - \Delta J \), where \( J_{\omega_o} \) is the FIM for \( \phi \) in the absence of the interference (the case of a diagonal noise covariance matrix) given by

\[
J_{\omega_o} = \frac{M^2}{N^2(1+M \rho^2) r^2 + 2M \rho^2 r^2} \tag{15b}
\]

and \( \Delta J \) is a non-negative term which increases the CRBL of \( \omega_o \) and is due to the presence of the interference.

\[
\Delta J = \frac{(2 \rho^2 r^2 M^{2}(1+M \rho^2))}{N^{2} \rho^2} \tag{18}
\]

Fig. 1: A nulling processor having output power \( Y \) due to an interference at bearing \( \beta \).
In (15)-(16) \( S(\omega) \) \( \left( \begin{array}{c} \omega \\ \omega \end{array} \right) \), \( \Phi(\omega) \) \( \left( \begin{array}{c} \omega \\ \omega \end{array} \right) \), \( R = R_1 + R_2 \) \( \left( \begin{array}{c} p & q \\ p & q \end{array} \right) \) \( \left( \begin{array}{c} m & n \end{array} \right) \) \( \left( \begin{array}{c} m & n \end{array} \right) \) \( \left( \begin{array}{c} p & q \\ p & q \end{array} \right) \) \( \left( \begin{array}{c} m & n \end{array} \right) \) \( \left( \begin{array}{c} p & q \\ p & q \end{array} \right) \) and \( Z I = \frac{2}{\pi} \).

It is now clear that the FIM (and therefore the CRLB) for the estimation errors of the spatial and spectral parameters of a source in the presence of coherence interference (with known parameters) can be put in terms of SNR and INR, the number of sensors \( M \), the conventional beam-pattern \( Z \) and its first derivative, the null pattern \( Y \) and the fundamental error \( J_0 \).

All of these factors have physical meaning: hence one can describe the asymptotic performance of the optimal estimator in simple physical terms. The following observations can be made:

a. The case of a diagonal noise covariance matrix is a special case of the one presented here. By setting \( \beta = 0 \) \( \left( \begin{array}{c} \beta \end{array} \right) \) the FIM \( J \) given in (15)-(16) reduces to the well-known result of spatially white noise. In particular, \( J_0 = 0 \) for \( \beta = 0 \), so the estimation errors of spatial and spectral parameters are uncorrelated. For \( \beta = 0 \) eq. (16) indicates that the key factors determine the geometrical coupling between spatial and spectral parameters is the BP derivative \( Z I = 2 \). Thus, for small \( \beta \) the BP \( J_0 = 0 \). However, it does not follow that small values of \( Z \) insure weak coupling. Hence one cannot argue that source and interference separated by more than a beamwidth insures weak coupling, even if there are no high sidelobes. What is required is that a beam steered on the interference exhibits a pattern which is essentially flat in the direction of the source. Whether its height at that point is large or small is of secondary importance.

b. The derivatives of \( Z \) do not appear in \( J_0 \) (Eq. (15)). For spectral estimation with known source bearing the height of the BP for a given source interference separation summarizes all relevant geometrical information. In particular, at nulls of the BP the CRLB on \( Z \) is the same as in the absence of the interference.

c. Denote the CRLB on \( \gamma \) for unknown \( S \) by CRLB(\( \gamma \)) and the bound on \( \gamma \) for known \( S \) by CRLB(\( \gamma \)/s) we have that

\[
\text{CRLB}(\gamma)/s = \frac{1}{M} x(\frac{1}{\lambda} J_0 - \lambda) \text{S}^{-1} \tag{19}
\]

\[
\text{CRLB}(\gamma) = \frac{1}{M} (J_0 - \lambda) \text{S}^{-1} \tag{20}
\]

It follows that unknown spectral parameters cause an incremental error in source bearing (related to \( \lambda J \)) which is not a function of \( ZI \). For \( \lambda J \) to be small, a beam steered on the source must carry little interference power. If the source spectral parameters are known, the improvement in the bearing estimator's performance is related to \( \lambda J \), which is a function of \( ZI \). This suggests that the knowledge of \( S \) can be used via the sensitivity of the beamformer output to variations in \( \lambda J \).

d. For large \( M \) and \( \lambda J \), \( R \) varies as \( M^2 \). It follows that \( J_0 \) and \( J_0 \) are not a function of \( M \) while \( J_0 \) varies as \( 1/M \). That is, when the source and interference are separated by more than a beamwidth, one can find \( M \) large enough so that the spectral and spatial parameters estimation errors are practically zero. Moreover, all \( J_0 \), \( J_0 \) and \( J_0 \) vary as \( 1/M \), \( 1/M^2 \), respectively so that the bounds on \( \lambda J \) in the presence and absence of interference do not differ markedly for separations in excess of a beamwidth.

e. \( \gamma \) has certain features regardless of array geometry:

Within unknown spectral parameters it has a single peak at \( 0\%/\lambda J \) where \( \lambda \) is the signal wavelength and \( L \) is the array diameter, \( \lambda J \) radians approximately the beamwidth of the mainlobe. That is, at spatial separation less than a beamwidth, the estimation accuracy is poor, with smaller estimation errors at \( \beta = 0 \) then at \( \beta = 0 \). With known spectral parameters the bound shows a main lobe near \( \beta = 0 \). Main lobes is much narrower than the BP width, suggesting that one can achieve much better resolution than with the conventional beamformer. This is the case without any unknown spectral parameters. Notice also the existence of a sidelobe of the bound within the BP mainlobe (having a peak at about the first peak of the null pattern). This sidelobe (which can be higher than the mainlobe for smaller interference to source ratios) was also observed by Heilweil (7).

3. NUMERICAL EXAMPLES

Consider the regular hexagonal array shown in Fig. 2. If the array is steered in the direction \( \beta = 0 \), Fig. 3a shows the beam pattern \( Z \) and its first derivative \( ZI \) (Fig. 3b) as a function of \( \beta \). In Fig. 3c the null pattern \( Y \) (eq. (14)) is depicted. Fig. 4 shows the CRLB(\( \gamma \)) and the CRLB(\( \gamma \)/s) for \( \beta = 0 \) as a function of \( \beta \). The source SNR is \( p = 1 \) and the interference INR is \( p = 20 \). Also shown in Fig. 4 the CRLB for \( \beta \) when the interference is absent, CRLB(\( \gamma \)/s) is also the dashed horizontal line. Fig. 5 gives equivalent results for an array of the same shape and dimension with \( M = 13 \) instead of \( M = 6 \). (One sensor was added in the center of the array of Fig. 2 and 6 more were added symmetrically on the hexagon edgers). By looking at Figs. 2 to 5 the following observations can be made:

a. As expected, CRLB(\( \gamma \)/s) is lower than CRLB(\( \gamma \)). Also, since \( \beta = 0 \), \( ZI = 0 \) and \( \beta = 0 \), the bounds are equal at \( \beta = 0 \).

b. CRLB(\( \gamma \)) is similar to CRLB(\( \gamma \)/s) in regions where \( ZI \) is small (\( 0^\circ \leq \beta \leq 50^\circ \) and \( 120^\circ \leq \beta \leq 140^\circ \)). This confirms the result that with small \( ZI \) the estimation errors of spectral and spatial parameters are uncorrelated. Notice, however, that with large \( ZI \) the difference between the CRLB(\( \gamma \)) and the CRLB(\( \gamma \)/s) is high as 13dB.

c. In the presence of interference the bound is as much as 12-13dB higher then when the interference is absent. The CRLB(\( \gamma \)) is similar to CRLB(\( \gamma \)/s) only in regions of small \( ZI \), small \( \beta \) and small \( ZI \) (\( \beta = 36^\circ \), \( 80^\circ \leq \beta \leq 110^\circ \)).

d. Figs. 4 and 5 show similar behavior of the bounds within the mainlobe of the BP. However, for \( \beta = 30^\circ \), CRLB(\( \gamma \)/s) is lower than CRLB(\( \gamma \))/s with \( M = 13 \) (Fig. 5) only. This confirms our contention that for large \( M \) and separations larger then a beamwidth the presence of interference has little influence on the bound.

e. Within the mainlobe of the BP (\( \beta = 0 \)) the bound on \( \beta \) has certain features regardless of array geometry:

With unknown spectral parameters it has a single peak at \( 0\%/\lambda J \) where \( \lambda \) is the signal wavelength and \( L \) is the array diameter, \( \lambda J \) radians approximately the beamwidth of the mainlobe. That is, at spatial separation less than a beamwidth, the estimation accuracy is poor, with smaller estimation errors at \( \beta = 0 \) then at \( \beta = 0 \). With known spectral parameters the bound shows a main lobe near \( \beta = 0 \). Main lobe is much narrower than the BP width, suggesting that one can achieve much better resolution than with the conventional beamformer. This is the case with unknown spectral parameters. Notice also the existence of a sidelobe of the bound within the BP mainlobe (having a peak at about the first peak of the null pattern). This sidelobe (which can be higher than the mainlobe for smaller interference to source ratios) was also observed by Heilweil (7).
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Fig. 4 : The Cramer-Rao bounds, M=6.

Fig. 5 : The Cramer-Rao bounds, M=13.

4. REFERENCES


