THE METHOD OF PROJECTIONS ONTO CONVEX SETS (POCS) FOR RESTORING IMAGES FROM INCOMPLETE INFORMATION

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RESUME

Dans cet article on utilise la méthode de projection sur les ensembles convexes (appelé POCS) pour restaurer les images à partir de l'information incomplète. La méthode POCS est équivalente à trouver un point de l'intersection de m ensembles convexes, où m est le nombre de propriétés de l'image connues à priori. On démontre comment la méthode POCS est utilisée pour restaurer les images à partir de 1. l'information incomplète dans l'espace de Fourier et 2. l'information dans l'espace de phase seulement.

SUMMARY

In this paper, we apply the method of projections onto convex sets (POCS) for restoring images from incomplete information. The method of POCS is equivalent to finding a point that lies at the intersection of m convex sets where m is the number of a priori-known properties of the image. We shall show how POCS is used in restoring images from 1. incomplete Fourier-space information and 2. phase information only.
THE METHOD OF PROJECTIONS ONTO CONVEX SETS (POCS)
FOR RECONSTRUCTING IMAGES FROM INCOMPLETE INFORMATION

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INTRODUCTION

In several important practical situations it is required to restore a signal or image from incomplete information. The problem of reconstructing a tomographic image from partial view data is one of them. Besides the well-known medical applications in computed tomography (CT) there is also a wide range of nonmedical applications in meteorology, electron microscopy, geophysics, astronomy, and oceanography which require construction of an image from partial information. For this reason Gerchberg-Papoulis (GP) algorithms [2], [3], [6], and their offsprings [4], [5] have generated considerable interest.

The method of projections onto convex sets (POCS) [1] has a significant advantage over the GP and related algorithms in that it enables a large number of a priori known constraints to be incorporated in the algorithm provided that they are formulated as constraints that restrict the image function to lie in a closed convex set. Not every constraint can be viewed as a restriction in terms of convex sets. For example the operation of digitizing a signal is not equivalent to projection onto a convex set. However numerous other constraints can be treated as convex set restrictions and we shall illustrate this with examples. Eleven signal and image constraints and their associated projection operators are furnished in [7] and many of those are of physical significance.

METHOD OF PROJECTIONS ONTO CONVEX SETS (POCS)

The basic idea in POCS is the following: Every a priori known property of an unknown image function f ∈ ℋ (ℋ a Hilbert space) is viewed as a constraint that restricts the signal to lie in a well known closed convex set C. Thus for m properties, there are m sets \( C_1, C_2, \ldots, C_m \) and the function f must lie in the intersection \( C_0 = \cap_{i=1}^{m} C_i \). The problem is then to find a point of \( C_0 \) given the sets \( C_i \) and the operators \( P_i, i = 1, 2, \ldots, m \) that project onto \( C_i \). Given an arbitrary \( f \), its projection onto \( C_1 \) is that element \( h \) that satisfies

\[
\min_{y \in C_1} ||f-y|| = ||f-h||. \tag{1}
\]

The restoration algorithm that "finds" a point of \( C_0 \) has the form

\[
f_{k+1} = P_m P_{m-1} \ldots P_1 f_k \quad k = 0, 1, 2, \ldots \tag{2}
\]

with \( f_0 \) arbitrary. More generally, we can write

\[
f_{k+1} = P_m P_{m-1} \ldots P_{i+1} f_i \quad k = 0, 1, 2, \ldots \tag{3}
\]

where \( P_i = 1_{\text{ext}}(P_i-L) \) and \( 0 < \lambda_i < 1 \). The relaxation parameter \( \lambda_i \) can be used to accelerate the rate of convergence, both initially and in the vicinity of solution. The theoretical basis of the algorithm is given in [7], [10]. In general convergence to an \( \text{ROF} \) is weak.

Because we shall deal with space-limited objects, defined by their reflectance, transmittance, or absorbance over a region \( \Omega \), the functions of interest in this study will be assumed to be members of \( L^2(\Omega) \), the space of all functions \( f(x,y) \) square-integrable over \( \Omega \). The associated Hilbert space \( \mathcal{H} \) is \( L^2(\Omega) \) with inner product and norm defined by, respectively,

\[
\langle g, h \rangle_\mathcal{H} = \int g(x,y)h(x,y) \, dx \, dy \tag{4}
\]

\[
||g||_\mathcal{H} = \left( \int g(x,y)^2 \, dx \, dy \right)^{1/2} \tag{5}
\]

In this paper we briefly demonstrate the application of the method of POCS to 1) reconstruction in CT from partial view data (limited angular view problem) and 2) reconstruction of the magnitude of the Fourier transform from phase.

APPLICATION OF POCS IN CT

In CT by Direct Fourier reconstruction method (DFR), each view gives the projection data from which a single central slice of the discrete Fourier transform (DFT) is obtained. When all the views are obtained, the Fourier transform plane is packed with the Fourier data on a polar raster. After interpolation to a cartesian format as described in [8], an inverse 2-D DFT is computed which yields the desired image. But in the case of incomplete view data the known Fourier data will be in a data cone with a subtended angle of \( < 180^\circ \). The following sets and associated projection operators express some a priori known properties of the image function. These are significant in CT.

1. \( C_1 \): The set of all functions that vanish outside a prescribed region \( \Omega \). Given an arbitrary \( f \in \mathcal{H} \), its projection onto \( C_1 \) is realized by

\[
P_1 f = P_1 f = f \quad 0, (x,y) \in \Omega \tag{6}
\]

2. \( C_2 \): The set of all functions in \( \mathcal{H} \) whose Fourier transforms assume a prescribed value \( \mathcal{G} \) over a closed region \( L \) in the \( u-v \) Fourier plane. The projection of an arbitrary \( f \) onto \( C_2 \) is realized by

\[
P_2 f = P_2 f = \{ u(v) \} \quad P_{u,v} \{ u(v) \} = \mathcal{G} \tag{7}
\]

where \( P_{u,v} = \{ f(x,y) \} \) etc. for \( u, v \) and \( F \) is the Fourier transform operator. In particular \( G(u,v) \) is the known portion of the spectrum of \( f \) over the data cone.

3. \( C_3 \): The set of all nonnegative real functions in \( \mathcal{H} \) that satisfy the energy constraint

\[
\int_{\Omega} |f(x,y)|^2 \, dx \, dy \leq E \tag{8}
\]

The projection of an arbitrary \( f \) onto \( C_3 \) is realized by

\[
P_3 f = P_3 f = \begin{cases} f^* & \text{if } f^* \leq E \\ F_{1,0} f^* & \text{if } f^* > E \end{cases} \tag{9}
\]

where \( f^* \) is the real part of \( f \), \( f^* \) is the rectified portion of \( f^* \), and \( E_{1,0} \) is the energy in \( f^* \); i.e.,

\[
E_{1,0} = \int_{\Omega} |f(x,y)|^2 \, dx \, dy \tag{10}
\]

4. \( C_4 \): The subset of all functions in \( \mathcal{H} \) that are nonnegative. The projection of an arbitrary \( f \) onto \( C_4 \) is realized by

\[
P_4 f = P_4 f = f \quad f \geq 0 \tag{11}
\]
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\[ P_2 f = \begin{cases} f_1 f_2 > 0 & (f_1 + f_2) \\ 0, \\ \text{otherwise.} \end{cases} \]  

5. \( \mathbb{C}_2 \): The set of all real functions in \( f \) whose amplitudes must lie in a prescribed closed interval \([a, b] \) \( a \geq 0, b \geq 0, a \leq b \). The projection onto \( \mathbb{C}_2 \) is realized by the following rule:

\[ P_2 f = \begin{cases} f_1(x,y) < a & \\ f_1(x,y) \geq a \leq b & \\ f_1(x,y) > b. \end{cases} \]  

In the above, \( f \) is \( f_1 + f_2 \).

Simulation of Limited Angular View Problem

The simulated image consists of three nested rectangles the longest being 24x32 pixels, centered on a 64x64 pixel field; it is shown in Fig. 1. The object is represented by a sequence \( f(m,n) \), \( m = 1, 2, \ldots, N, n = 1, \ldots, N(N = 64) \) which represents the gray levels of the \( mn \)th pixel. The gray levels are confined to \( 0 \leq f(m,n) \leq 1 \) and are 0.4, 0.8, and 1.0 in going from the largest to the smallest rectangle, respectively. The background level is zero.

![Fig. 1. The image: it consists of three nested rectangles of gray levels 1.0 (center), 0.8 and 0.4.](image)

In our simulation a 90 degree data cone is used in all the experiments. We now summarize some a priori known facts and assumptions about the image.

A Priori Constraints

1) Image support confined to rectangular region \( \Delta x = 55 \) pixels, \( \Delta y = 57 \) pixels.

2) Gray levels \( f \) satisfy \( 0 \leq f \leq 1 \).

3) Energy over 6x64 pixel field cannot exceed \( \rho = 268.5 \).

4) Initially known spectrum \( \phi(u,v) \) spectrum of the \( \phi(u,v) \) image is \( \phi(u,v) \).

\[ (\chi_0(\phi) \text{ is } 90^\circ \text{ cone}). \]

The formula used for computing energy is

\[ \sum_{m=0}^{M} \sum_{n=1}^{N} f^2(m,n) = 267. \]

Algorithms

The following algorithms are implemented by a flexible program called PROCON whose description is given in [9].

1) Gerchberg-Papoulis (GP): The familiar G-P algorithm can be written in compact form, for our problem, as

\[ f_{k+1} = P_{I_k} f_k. \]

2) UNIRELAX 1: \( f_{k+1} = P_{I_k} f_k \)

3) RELAX 1: \( f_{k+1} = \frac{f_k + T f_k}{2} \) with \( \lambda_2 = 1.75, \lambda_3 = 1.9995 \).

Results

Figure 2 shows the reconstruction error as a function of iteration number \( k \) for GP, UNIRELAX 1 and RELAX 1, using the method of POCS significantly outperform the GP algorithm.

![Fig. 2. Error versus iteration number \( k \) for GP, UNIRELAX 1, and RELAX 1.](image)

In Fig. 3 are shown the actual restorations from the 90 degree data cone after 30 iterations.

Figure 3 furnishes impressive evidence that methods based on projections onto convex sets furnish markedly superior restorations to the Gerchberg-Papoulis method.

We note that projections onto \( C_0 \) and \( C_1 \) do not appear in Eqs. (1h) or (1s). First of all, since \( C_0 \) \( C_1 \), it follows that \( P_1 \) (or \( T_1 \) would be more likely) to restrict the solution set than \( P_0 \) (or \( T_0 \)). However, when \( P_1 \) (or \( T_1 \) was actually used its affect was negligible so that we ignored it in the results presented here.
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Fig. 3. Restoration of the image. Counterclockwise from upper left: original object, UNIRELAX 1, GR, RELAX 1.

RESTORATION FROM PHASE (RFP) BY POCS

The two sets of principal interest in the restoration-from-phase (RFP) problem are

\[ C_1 = \{ f(x) : f(x) = 0, |x| > a \} \]
\[ C_2 = \{ f(x) : \arg(\hat{f}(\omega)) = \phi(\omega) \}. \]

(16)

(17)

In words: \( C_1 \) is the set of space-truncated functions and \( C_2 \) is the set of all \( f \) real-valued in \( H \) with prescribed phase. Both \( C_1 \) and \( C_2 \) are closed convex sets. With \( P_1 \) denoting the projection operator onto \( C_1 \), it is not difficult to show that \( P_1 \) and \( P_2 \) are realized by

\[ P_1 f = \begin{cases} \frac{f(x)}{|x|}, & |x| \leq a \\ 0, & |x| > a \end{cases} \]

(18)

and

\[ P_2 f = \left( \frac{\omega}{|\omega|} e^{j\phi(\omega)} \right) \hat{f}(\omega), \quad \omega \in \Omega \]

(19)

where \( \phi(\omega) \) is the prescribed phase, \( f(x) \) is an arbitrary element of \( H \) and \( \hat{f}(\omega) = \hat{f}(\omega) \) is the Fourier transform of \( f(x) \). \( \Omega \) is the set given by

\[ \Omega = \{ \omega : \cos(\phi(\omega) - \psi(\omega)) \geq 0 \} \]

(20)

and \( \Omega^c \) by

\[ \Omega^c = \{ \omega : \cos(\phi(\omega) - \psi(\omega)) < 0 \} \]

(21)

i.e., the complement of \( \Omega \). As defined, \( P_2 \) is a non-linear operator while \( P_1 \) is linear. From now on we shall use the following notation:

\[ f(x) \leftrightarrow \hat{f}(\omega) = \int_{-a}^{a} f(x) e^{-j2\pi \omega x} dx \]

(22)

We consider the per-cycle optimization we mean the following: For a given \( f \) find \( \lambda_1^m \) and \( \lambda_2^m \) such that \( |e_{n+1}^m| \) is minimum. This method is described in detail in [11]. In the general case when \( P_2 \) is a non-linear operator \( \lambda_1^m \) can be approximated by 1) finding an approximate expression for the error \( |e_{n+1}^m| \) or, 2) minimizing the expression

\[ \lambda_1^2 \approx \lambda_1^2 \approx \frac{|P_1 f_{n+1} - \hat{f}_{n+1}|^2}{|P_2 f_{n+1} - \hat{f}_{n+1}|^2} \]

(23)

which is equivalent to minimizing some upper bound of \( |e_{n+1}^m| \). This minimization can be done by a straightforward scan through the range of \( \lambda_1 \). Then, having obtained an approximate \( \lambda_1^m \), we compute \( \lambda_2^m \) from

\[ \lambda_2^m = \frac{\text{Re}([P_1 T_{\lambda_1} f_{n+1} - P_2 T_{\lambda_1} f_{n+1}])}{|P_1 T_{\lambda_2} f_{n+1} - P_2 T_{\lambda_2} f_{n+1}|^2} \]

(24)

where \( (x,y) \) for any \( x,y \in H \) means the inner product in the Hilbert space \( H \). Since \( f_1 \) is unknown we use \( f_1 \) to calculate the unknown part of \( f \). A similar technique is used when \( \lambda_2^m \) is determined by \( \lambda_1 \). In the case where \( P_2 \) is a linear operator we can show that \( \lambda_2^m \) and \( \lambda_1^m \) are related and we can obtain a closed form solution for \( \lambda_1^m \).

We consider two possible computations for the RFP problem:

\[ \mathbf{a:} \quad f_{n+1} = P_1 T_{\lambda_1} f_n \]

(25)

In this case the last operator in the cycle is linear and it can be shown [11] that

\[ \lambda_1^m = 1 \]

(26)

and

\[ \lambda_2^m = \frac{|P_1 f_{n+1} - P_2 f_{n+1}|^2}{|P_1 f_{n+1} - P_2 f_{n+1}|^2} \]

(27)

(28)

the last term on the right in Eq. (26) is always positive and can be either 1) approximated by using its Fourier transform domain equivalent form and replacing \( \hat{f}(\omega) \) by \( |\hat{f}(\omega)| \), or 2) neglected. When the last term on the right of Eq. (26) is neglected the resulting result \( \lambda_2^m \) is actually a lower bound. Results using this lower bound (subject to it not exceeding the value of 2) is given in the example.

**EXAMPLE**

The example given in this paper shows the dependence of the restoration on the initialization, the number of iterations and on the optimization of the \( \lambda \)'s. The original signal to be restored from phase-only is given in Fig. 1 and the results for the different cases are summarized in tables 1 and 2. The error given in the tables is a per-cent error defined by
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\[ e_n = 100 \cdot \frac{|r_n - r|}{||r||} \]  

(27)

The signal to be restored is a truncated 128 point cosine on a pedestal given by

\[ f(x) = \begin{cases} 
0.5 + 0.5 \cos(\pi x/30), x=1, \ldots, 50 \\
0, x=51, \ldots, 128 
\end{cases} \]  

(28)

In tables 1 and 2 \( F \) denotes the initial arbitrary value of the Fourier transform magnitude (this is an input to the program furnished by the user). Table 1 gives the results for \( P=10 \), and table 2 for \( P=10 \exp(-\omega^2/100) \), where \( \omega=0, 1, 2, \ldots \) (i.e., is discrete). The main conclusions drawn from these tables are:

- The method of POCS effectively restores the signal even with no optimization of the \( \lambda \)'s.
- The results depend strongly on the initialization \( P_0 \).
- Per-cycle optimization of the \( \lambda \)'s significantly improves restoration over pure projections \( \lambda_1=\lambda_2=1 \). For the same final error and with no optimization we need at least twice the number of iterations that we need with optimization.
- The order in which the operators are applied does affect the rapidity of convergence. A "best" ordering procedure is not, at present, known.
- Because of the importance of good initialization, one should attempt to obtain or use a priori information that will get \( P \) as close to \( P_0 \) as possible.

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<th>Optimization for ( f_{n+1} = T_2 f_n )</th>
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**CONCLUSIONS**

In this paper we discussed the applications of the method of POCS to 1) CT image restoration and 2) the restoration-from-phase problem. In general the method allows for any number of a priori known image constraints to be incorporated in the algorithm provided that these can be associated with convex sets. We discussed methods of approximately optimizing the relaxation parameters and showed thereby that a significant improvement in performance can be obtained. This algorithm has the property of guaranteed convergence (strong convergence in the finite-dimensional case) with and without the use of relaxation parameters.

**ACKNOWLEDGEMENTS**

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**REFERENCES**


