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La Detection et Estimation d'une Impulsion en Utilisant  
la Transformation Discrete de Fourier

The Detection and Estimation of a Single Impulse Using  
the Discrete Fourier Transform

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## RESUME

Etant donné un vecteur  $\underline{x} = (x[0], x[1], x[2], \dots, x[N-1])$ , dont les éléments  $x[i]$  sont des nombres complexes et que la transformation Discrete de Fourier (D.F.T.) du vecteur  $\underline{x}$  à la propriété suivante: c'est à dire que deux éléments consécutifs égalent zero.

Premièrement nous montrons que si  $\underline{Y} = \underline{x} + I \delta_i$  ou  $\delta_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$  et  $I$  est un nombre complexe arbitraire, alors le vecteur original  $\underline{x}$  ne peut être déterminé qu'à partir du vecteur observé  $\underline{Y}$  (sans erreur). Puis nous examinons le cas où  $\underline{Y} = \underline{x} + I \delta_i + \underline{n}$  ou  $\underline{n}$  est un vecteur dont les éléments sont i.i.d.\* variables aleatoires Gaussiennes. Nous trouvons la règle de détermination optimale pour déterminer la localité de l'impulsion  $I$  (c'est à dire pour déterminer la valeur de  $i$ ). Finalement nous déterminons le rendement de ce système optimal.

\*i.i.d.: (indépendente équi-répartie.)

## SUMMARY

Given a vector  $\underline{x} = (x[0], x[1], \dots, x[N-1])$  whose components  $x[i]$  are complex numbers. Assume that the Discrete Fourier Transform (D.F.T.) of the vector  $\underline{x}$  has the property that two consecutive components are identically zero.

First we show that if  $\underline{y} = \underline{x} + I \delta_i$  where  $\delta_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$  and  $I$  is an arbitrary complex number, then the original vector  $\underline{x}$  can be uniquely determined from the observed vector  $\underline{y}$  (without error). Then we investigate the case where  $\underline{y} = \underline{x} + I \delta_i + \underline{n}$  where  $\underline{n}$  is a vector whose components are i.i.d. Gaussian random variables. We find the optimal decision rule for determining the location of the impulse  $I$  (that is, for determining the value of  $i$ ). Finally we determine the performance of this optimal system.



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### 1. Introduction

In this paper we explore a seemingly new method of locating and removing impulses which occur in a time-discrete but continuous-amplitude time series. It is based upon the Discrete Fourier Transform (D.F.T.) and principles previously utilized in algebraic coding theory.

The basic notion used in our system is as follows. We consider a discrete-time sequence whose D.F.T. has zeros in certain positions. After transmission over an additive impulse noise channel, the D.F.T. of the received discrete-time sequence no longer has zeros in those positions. Specifically, the values of the components of the received sequence's D.F.T. in these positions are due only to the noise on the channel. Assuming that the noise on the channel is impulsive in nature, a curve fitting strategy is used to find the location and values of the impulses that caused these spectral components. These impulses are then removed to result in a better estimate of the transmitted sequence.

It should be emphasized that our ideas are broader based than the D.F.T. description presented here. For example, if one passed our discrete-time sequence through an invertible nonlinearity the resulting sequence would not have zeros in the D.F.T. Yet if impulses were added to this sequence, our techniques would still apply since one could pass the resulting sequence through the inverse nonlinearity, and then proceed in the same manner as described in this paper.

The usual techniques for combating impulsive noise are to amplitude limit in the time domain and to linearly filter in the frequency domain. Amplitude limiting in the domain cannot be used for continuous amplitude signals since one might clip peaks of the desired signal. Linear filtering can remove the components of the noise that are "out-of-band" but is much less effective than the method suggested here. In fact, it is exactly the "out-of-band" noise removed by linear filtering that gives us the syndrome (or signature) of the impulses to be removed. Once the impulses are located by our method both their "in-band" and "out-of-band" components can be subtracted out.

### 2. General Theory

Let  $x[i]$ ,  $i=0,1,\dots,N-1$  be a data sequence over the field of complex numbers and let  $X[k]$ ,  $k=0,1,\dots,N-1$  be the spectral components of the DFT where  $X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk}$  and  $W_N$  is an  $N$ th root of unity. Let  $K$  be a positive integer less than  $N$ , and let "a" be any integer in the range  $0 < a < N-1$ . We assume that our data sequence has the special property that  $X[a] = X[a+1] = \dots = X[a+N-K-1] = 0$  where the indices are assumed to be taken modulo  $N$ . For this class of data sequences, it has been shown that [1]:

**Theorem 1:** The entire data sequence  $x[0], x[1], \dots, x[N-1]$  can be reconstructed from any  $K$  or more of its components.

**Theorem 2:** If  $y[i] = x[i] + e[i]$ ,  $i=0,1,\dots,N-1$ , where only  $T$  of the components of  $e[0], e[1], \dots, e[N-1]$  are non-zero, then  $x[i]$ ,  $i=0,1,\dots,N-1$  can be reconstructed from the sequence  $y[0], y[1], \dots, y[N-1]$  without error provided that  $T \leq (N-K)/2$ .

In this paper we restrict ourselves to the case of  $T=1$ , that is, the correction of a single impulse. Thus we require that  $(N-K) \geq 2$ . So that we may simplify the discussion we will assume that  $X[N-2] = X[N-1] = 0$  but any two consecutive zeros would do.

Let us assume that for  $T=1$ , the non-zero value of  $e[i]$  is  $e[m] = I$ . That is, we assume we have an impulse in the  $m$ -th position and that the value of this impulse is the complex number  $I$ . If we take the D.F.T. of the errored data sequence  $y[0], y[1], \dots, y[N-1]$  to result in the  $N$  spectral components  $Y[0], Y[1], \dots, Y[N-1]$  we find that since  $X[N-2] = X[N-1] = 0$ , then

$$Y[N-2] = IW_N^{m(N-2)}$$

and

$$Y[N-1] = IW_N^{m(N-1)}$$

We can solve these equations for  $I$  and  $m$  as follows:

$$m = -j \frac{N}{2\pi} \ln(Y[N-1]/Y[N-2]), \text{ modulo } N,$$

$$I = (Y[N-2])^{N-1} / (Y[N-1])^{N-2}.$$

Thus, we see that in the absence of other errors, a single (impulsive) error can be located and its value determined by examination of two components of the D.F.T. where the data sequence has zeros.

### 3. The Case of Random Noise

We now consider the more interesting case where the error sequence is the sum of a random error sequence and an impulse. In particular we consider the situation where  $y[j] = x(j) + n(j) + e(j)$ ,  $j=0, 1, \dots, N-1$  where  $n[0], n[1], \dots, n[N-1]$  is an independent identically distributed (i.i.d.) sequence of Gaussian random variables of mean 0 and variance  $\sigma^2$ . As before  $e(j)$  is a single impulse in the  $m$ th component of magnitude  $I$ . Since  $X[N-2] = X[N-1] = 0$  by assumption then we have

$$Y[n] = \sum_{i=0}^{N-1} (\text{Re}(n(i)) + j \text{Im}(n(i))) W_N^{in} + (\text{Re}(I) + j \text{Im}(I)) W_N^{mn} \quad n=N-2, N-1$$

Simplifying this expression we can write

$$Y[N-2] = Z[N-2] + IW_N^{-2m}$$

$$Y[N-1] = Z[N-1] + IW_N^{-m}$$

where  $Z[N-2]$  and  $Z[N-1]$  are sums of complex Gaussian random variables and are thus complex Gaussian random variables. Again, these equations are to be solved for  $I$  and  $m$ . Let us call our solutions  $\hat{m}$  and  $\hat{I}$ . Our objective is to find solutions such that:

- (a)  $\Pr\{\hat{m}=m\}$  is a maximum
- (b)  $E\{|I-\hat{I}|^2 \mid \{\hat{m}=m\}\}$  is a minimum.

It may be shown that if the phase of the impulse is uniformly distributed over the interval  $[0, 2\pi]$  and if the location of the impulse is chosen with equal probability from the set  $\{0, 1, \dots, N-1\}$  then the best estimate of the location of the impulse in the sense of minimizing the probability of error is

$$\hat{m} = \langle (\arg(Y[N-2]) - \arg(Y[N-1]))N/2\pi \rangle$$

where  $\langle \rangle$  is the nearest integer to the real number contained within the bracket. Furthermore, from [2], we know that the probability that  $\hat{m}$  is not equal to  $m$  is given as

$$P_r\{\hat{m} \neq m\} = \int_{-\pi/2}^{\pi/2} \frac{|I|^2 (1 - \cos(\pi/N) \cos t)}{2\pi \sigma^2 (1 - \cos(\pi/N) \cos t)} dt.$$

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Calculated values of this error probability are given in Figure 1 for the following codes:  $(N,K) = (4,2), (8,6), (16,14), (32,30)$  and  $(64,62)$ . The results of a computer simulation of the  $(4,2)$  code are also shown for comparison.

The value of the impulse was estimated using the following estimator

$$\hat{I} = \frac{1}{2} (W_N^{2\hat{m}} Y[n-2] + W_N^{\hat{m}} Y[N-1])$$

For the case where  $\hat{m}=m$ , we then find that

$$I = I + \text{Re}(n[m]) + j \text{Im}(n[m]) + \mathcal{N}$$

where both the real and imaginary parts of  $\mathcal{N}$  are Gaussian with mean 0 and variance  $((N/2)-1)\sigma^2$ . Writing the components of our final estimate of the data vector as  $\hat{x}[j]$ , where  $\hat{x}[j] = y[j]$  for  $j \neq \hat{m}$  and  $\hat{x}[j] = y[j] - \hat{I}$  for  $j = \hat{m}$ , we find that

$$E \left[ \sum_{j=0}^{N-1} | \hat{x}[j] - x[j] |^2 \mid (\hat{m}=m) \right] = (3N-4)\sigma^2.$$

Similar results can be obtained for the case where  $\hat{m} \neq m$ . We can show that  $(3N-4)\sigma^2$  serves as a lower bound to the unconditional mean-squared error between the transmitted and decoded vectors. An upper bound to this unconditional mean-squared error is  $(3N-4)\sigma^2 + 2 \max_r P_r[\hat{m} \neq m]$ .

#### 4. Discussion

One method of generating a discrete-time, continuous amplitude sequence with characteristics similar to those described here is to sample a continuous-time band-limited waveform at a rate faster than the Nyquist rate. For finite  $N$ , the  $N$  by  $N$  D.F.T. will not have components which are identically zero but the high frequency components will be small.

Alternatively, we could create a discrete-time, continuous amplitude sequence whose D.F.T. has true zeros by inserting redundant samples in an arbitrary sequence. Starting with  $K$  arbitrary data points, one can always insert  $(N-K)$  additional data points so as to force  $(N-K)$  consecutive zeros in an  $N$  by  $N$  D.F.T.

This paper only considered the case of the detection and correction of single impulses. If  $T \leq [(N-K)/2]$  one can use a variant of a B.C.H. decoding algorithm to correct  $T$  or fewer impulses. Furthermore a "voting" strategy can be used for the case where  $T \leq (N-K-1)$ . These cases are discussed further in reference [1].

#### References

- [1] J.K. Wolf, "Redundancy, the Discrete Fourier Transform and Impulse Noise Cancellation," IEEE Transactions on Communications, to be published, 1983.
- [2] R.F. Pawula, S.O. Rice and J.H. Roberts, "Distribution of the Phase Angle Between Two Vectors Perturbed by Gaussian Noise," IEEE Transactions on Communications, Vol. COM-30, pp. 1828-1841, August, 1982.

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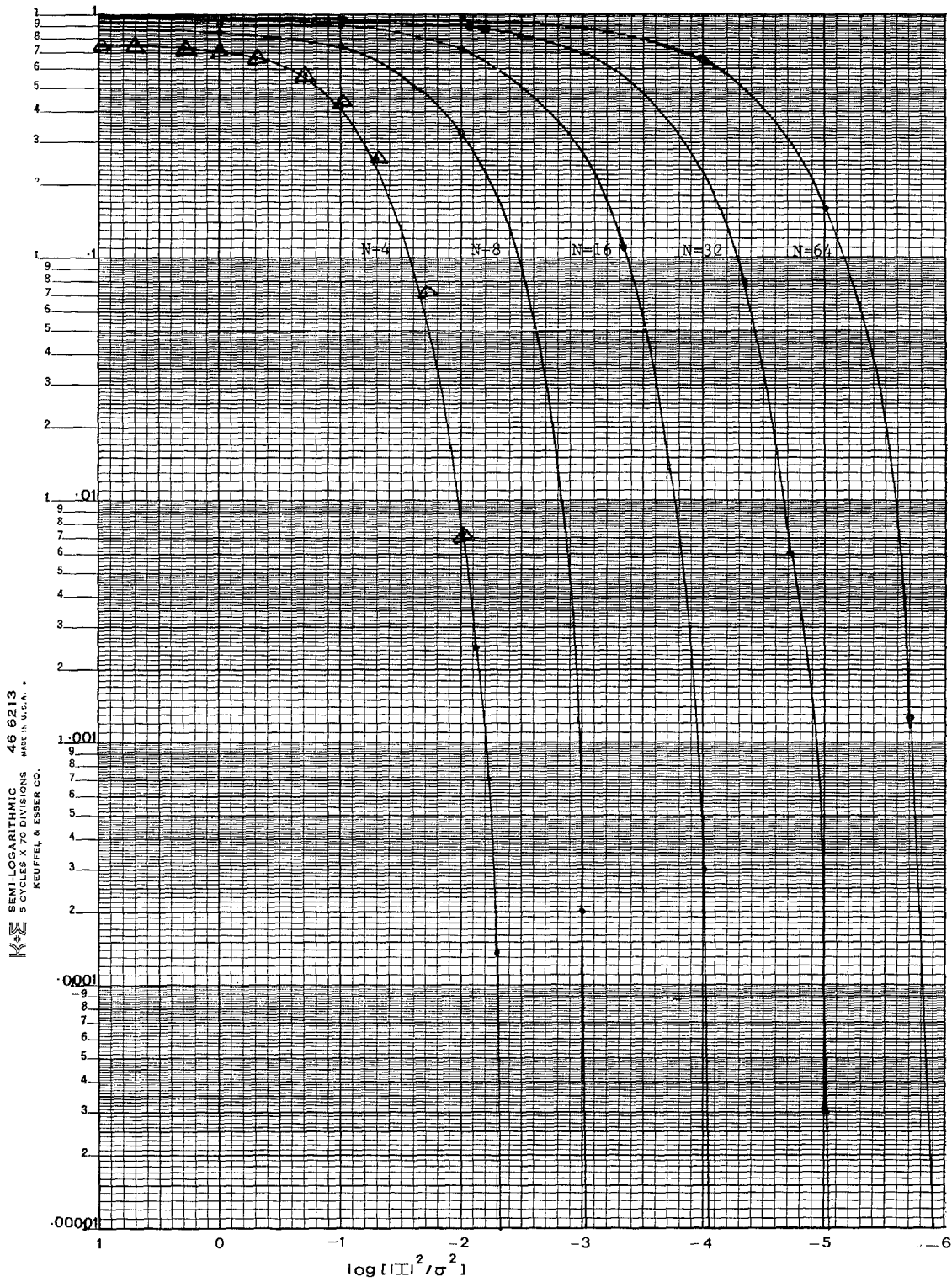


Figure 1. Probability of Error in Location of Impulses for (N,N-2) codes with N=4,8,16,32,64. Triangles Indicate simulation results for N=4 case.